Algebraic Product Is the Only "And-Like"-Operation for Which Normalized Intersection Is Associative: A Proof

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1. Fuzzy sets and normalized fuzzy sets: a brief reminder

- A *fuzzy set* on a universal set X is a function $\mu(x)$ that assigns:
  - to each element $x \in X$,
  - a number from the interval $[0, 1]$.
- Fuzzy sets were invented to describe imprecise ("fuzzy") natural-language properties such as "small".
- The value $\mu(x)$ is then a degree to which, according to the user, the object x has the desired property (e.g., is small).
- The degree 1 means that x definitely has the property.
- The degree 0 means that x definitely does not have this property.
- Intermediate value $\mu(x)$ correspond to x having the property "to some degree".
- Usually, an additional condition is imposed that the fuzzy set should be *normalized*, i.e., that $\sup\limits_{x \in X} \mu(x) = 1$.

2. "And"-operations (t-norms): a brief reminder

- In many cases:
  - we know the degrees a and b to which the properties A and B are satisfied, and
  - we want to use this information to estimate the degree to which the property "A and B" is satisfied.
- The function $f_\&(a,b)$ that generates the resulting estimate is known:
  - as an *"and"-operation*,
  - or, for historical reason, a *t-norm*.
- This function have several natural properties.
- For example, $A\,\&\,B$ means the same as $B\,\&\,A$.
- So the corresponding estimates for $A\,\&\,B$ and $B\,\&\,A$ should coincide, i.e., we should have $f_\&(a,b)=f_\&(b,a)$.
- In mathematical terms, the "and"-operation should be commutative.

3. "And"-operations (t-norms): a brief reminder (cont-d)

- Similarly, $(A \,\&\, B) \,\&\, C$ means the same as $A \,\&\, (B \,\&\, C)$.
- So, we should always have $f_\&(f_\&(a,b),c) = f_\&(a, f_\&(b,c))$.
- In mathematical terms, the "and"-operation should be associative.
- Also, intuitively:
  - if A is absolutely true. i.e., if $a = 1$,
  - then our degree of confidence in $A \,\&\, B$ is equal to our degree of confidence in B.
- In precise terms, this means that we should have $f_\&(1,b) = b$ for all b.
- Due to commutativity, we also have $f_\&(a,1) = a$ for all a.

4. Intersection and normalized intersection of fuzzy sets

- Suppose that we have two properties – as described by the sets S_1 and S_2 of all the elements that satisfy the corresponding property.
- Then:
  - the set of all the objects that satisfy the first property *and* satisfy the second property
  - is called the *intersection* $S_1 \cap S_2$ of the two sets.
- In the fuzzy case:
  - by the meaning of the fuzzy set,
  - the degrees to which an object x satisfies each of the properties are equal to $\mu_1(x)$ and $\mu_2(x)$.

5. Intersection and normalized intersection of fuzzy sets (cont-d)

- Thus, by the meaning of the "and"-operation:
  - the degree to which the first property is satisfied *and* the second property is satisfied
  - is equal to $f_\&(\mu_1(x), \mu_2(x))$ for an appropriate "and"-operation $f_\&(a, b)$.
- So, it is reasonable to define the intersection of two fuzzy sets as

$$\mu(x) \stackrel{\text{def}}{=} f_\&(\mu_1(x), \mu_2(x)).$$

- The problem with this definition is that:
  - if we, as usual, limit ourselves to normalized sets,
  - then the above definition may lead to a non-normalized set.
- For example, for some $a < 1$, on the universal set $X = \{1, 2\}$, let us consider two functions

$$f_1(1) = a, \quad f_1(2) = 1, \quad f_2(1) = 1, \quad f_2(2) = a.$$

6. Intersection and normalized intersection of fuzzy sets (cont-d)

- Then, in view of the fact that $f_\&(a, 1) = f_\&(1, a) = a$, we get $\mu(1) = \mu(2) = a < 1$, so $\max(\mu(x)) = a < 1$.
- If want the intersection to be a normalized fuzzy set:
  - we must *normalize* the expression,
  - i.e., divide it by the supremum of this expression:

$$(\mu_1 \,\&\, \mu_2)(x) = \frac{f_\&(\mu_1(x), \mu_2(x))}{\sup\limits_{y \in X} f_\&(\mu_1(y), \mu_2(y))}.$$

- It is reasonable to call this operation *normalized intersection.*
- *Comment.* The intersection is only defined when the denominator is not equal to 0.

7. When is normalized intersection associative?

- By definition, normalized intersection operation is commutative.
- It is known that for algebraic product $f_\&(a, b) = a \cdot b$, this operation is associative.
- For many other "and"-operations, normalized intersection is not associative.
- So, a natural conjecture emerged that:
  - the algebraic product is the only "and"-operation
  - for which the normalized intersection is associative.
- In this paper, we prove that the algebraic product is indeed the only "and"-operation for which the normalized intersection is associative.
- Moreover, we prove it:
  - not only for all "and"-operations,
  - but also for more general binary operations that are not necessarily commutative or associative.

8. Definitions

- *By an* "and"-like *operation, we mean a function* $t : [0,1] \times [0,1] \mapsto [0,1]$ *for which* $t(a,1) = t(1,a) = a$ *for all* $a \in [0,1]$.
- This definition is weaker than that of a t-norm.
- In particular, we do not require monotonicity, or even associativity.
- *For each "and"-like operation* $t(a,b)$*, the corresponding* normalized intersection *operation transforms:*
  - *two fuzzy sets* $\mu_1(x)$ *and* $\mu_2(x)$
  - *into a new fuzzy set* $(\mu_1 \,\&\, \mu_2)(x) = \dfrac{t(\mu_1(x), \mu_2(x))}{\sup\limits_{y \in X} t(\mu_1(y), \mu_2(y))}$.
- *Comment.* The intersection is only defined when the denominator of this formula is different from 0.

9. Main result

For each "and"-like operation $t(a, b)$, the following two conditions are equivalent:

- $t(a, b)$ *is the algebraic product, i.e.,* $t(a, b) = a \cdot b$*;*
- *The normalized intersection operation corresponding to* $t(a, v)$ *is associative, i.e., for all possible normalized fuzzy sets* μ_1*,* μ_2*, and* μ_3*:*
  - *the fuzzy sets* $(\mu_1 \cap \mu_2) \cap \mu_3$ *and* $\mu_1 \cap (\mu_2 \cap \mu_3)$ *are either both defined or both undefined, and*
  - *if they are both defined, then* $(\mu_1 \cap \mu_2) \cap \mu_3 = \mu_1 \cap (\mu_2 \cap \mu_3)$*.*

10. Discussion

- In this formulation, we only use two properties out of many usual properties of the "and"-operation (t-norm):
  - that for every a, we have $t(1, a) = a$, and
  - that for every a, we have $t(a, 1) = a$.
- From the mathematical viewpoint, a natural question is:
  - can we go even further, keep only one of these properties,
  - and still keep our result?
- The following two simple examples show that:
  - both above properties are needed,
  - one is not enough to conclude that $t(a, b) = a \cdot b$.
- The function $t(a, b) = b$ has the property that $t(1, a) = a$ for all a.
- For this function, as one can easily check, the normalized intersection of μ_1 and μ_2 is simply μ_2: $\mu_1 \cap \mu_2 = \mu_2$.

11. Discussion (cont-d)

- Thus $(\mu_1 \cap \mu_2) \cap \mu_3 = \mu_1 \cap (\mu_2 \cap \mu_3) = \mu_3$.
- The function $t(a, b) = a$ has the property that $t(a, 1) = a$ for all a.
- For this function, as one can easily check, the normalized intersection of μ_1 and μ_2 is simply μ_1: $\mu_1 \cap \mu_2 = \mu_1$, thus

$$(\mu_1 \cap \mu_2) \cap \mu_3 = \mu_1 \cap (\mu_2 \cap \mu_3) = \mu_1.$$

12. Proof, Part 1

- It is known that the normalized intersection operation corresponding to algebraic product is associative.
- So, to complete the proof, it is sufficient to prove that for each "and"-like operation $t(a, b)$:
  - if the corresponding normalized intersection is associative,
  - then $t(a, b) = a \cdot b$.

13. Proof, Part 2

- Indeed, assume that for an "and"-like operation $t(a,b)$, the corresponding normalized intersection is associative.
- Let us prove that in this case, $t(a,b)=a\cdot b$.
- If $b=1$, then, by definition of the "and"-like operation, we have $t(a,b)=t(a,1)=a$ and $a\cdot b=a\cdot 1=a$.
- So in this case indeed $t(a,b)=a\cdot b$.
- Thus, to complete the proof, it is sufficient to consider the case when $b<1$.

14. Proof, Part 3

- Let $a, b \in [0,1]$ be two numbers for which $b < 1$.
- Let us consider the following three normalized fuzzy sets on the universal set $X = \{1, 2\}$:

  $$\mu_1(1) = a, \;\; \mu_1(2) = 1, \;\; \mu_2(1) = 1, \;\; \mu_2(2) = a, \;\; \mu_3(1) = 1, \;\; \mu_3(2) = b.$$

- Let us use associativity to prove that $t(a, b) = a \cdot b$.
- Let us first compute $(\mu_1 \cap \mu_2) \cap \mu_3$.
- First, for $\mu_1 \cap \mu_2$, by the definition of the "and"-like operation, we get

  $$t(\mu_1(1), \mu_2(1)) = t(a, 1) = a, \;\; t(\mu_1(2), \mu_2(2)) = t(1, a) = a.$$

- Thus, $\max(t(\mu_1(1), \mu_2(1)), t(\mu_1(2), \mu_2(2)) = \max(a, a) = a$.
- In this case, if $a > 0$, then we get

  $$(\mu_1 \cap \mu_2)(1) = (\mu_1 \cap \mu_2)(2) = \frac{a}{a} = 1.$$

- (The case when $a = 0$ is considered in Part 4 of this proof.)

15. Proof, Part 3 (cont-d)

- Then, for $(\mu_1 \cap \mu_2) \cap \mu_3$, by the same definition of the "and"-like operator, we get:

$$t((\mu_1 \cap \mu_2)(1), \mu_3(1)) = t(1,1) = 1, \;\; t((\mu_1 \cap \mu_2)(2), \mu_3(2)) = t(1,b) = b.$$

- Thus $\max(t((\mu_1 \cap \mu_2)(1), t((\mu_1 \cap \mu_2)(2)) = \max(1,b) = 1$.
- Therefore, by the definition of the normalized interaction, we have:

$$((\mu_1 \cap \mu_2) \cap \mu_3)(1) = \frac{1}{1} = 1, \;\; ((\mu_1 \cap \mu_2) \cap \mu_3)(2) = \frac{b}{1} = 1.$$

- Let us now compute $\mu_1 \cap (\mu_2 \cap \mu_3)$.
- First, from the definition of the "and"-like operation, we get:

$$t(\mu_2(1), \mu_3(1)) = t(1,1) = 1, \;\; t(\mu_2(2), \mu_3(2)) = t(a,b).$$

- Here, $t(a,b) \le 1$ – since all the values of the function t are from the interval $[0,1]$.

16. Proof, Part 3 (cont-d)

- Thus:

$$\max(t(\mu_2(1), \mu_3(1)), t(\mu_2(2), \mu_2(2))) = \max(1, t(a, b)) = 1.$$

- Therefore:

$$(\mu_2 \cap \mu_3)(1) = \frac{1}{1} = 1, \quad (\mu_2 \cap \mu_3)(2) = \frac{t(a,b)}{1} = t(a, b).$$

- Now, from the definition of the "and"-like operation, we get

$$t(\mu_1(1), t(\mu_2(1), \mu_3(1))) = t(a, 1) = a,$$

$$t(\mu_1(2), t(\mu_2(2), \mu_3(2))) = t(1, t(a, b)) = t(a, b).$$

- According to the formula for the normalized intersection:
  - to find the values of the fuzzy set $\mu_1 \cap (\mu_2 \cap \mu_3)$,
  - we need to find the maximum of the above two values.
- The value of this maximum depends on which of the two values is larger; so let us consider both possible cases.

17. Proof, Part 3.1

- If $a \le t(a,b)$, then:

$$\max(t(\mu_1(1), t(\mu_2(1), \mu_3(1))), t(\mu_1(2), t(\mu_2(2), \mu_3(2))) = \max(a, t(a,b)) = t(a,b)$$

- Thus, we get:

$$(\mu_1 \cap (\mu_2 \cap \mu_3))(1) = \frac{a}{t(a,b)}, \quad (\mu_1 \cap (\mu_2 \cap \mu_3))(2) = \frac{t(a,b)}{t(a,b)} = 1.$$

- By associativity, these values should be equal to the values for

$$(\mu_1 \cap \mu_2) \cap \mu_3.$$

- By comparing the values of these two fuzzy sets for $x = 2$, we conclude that $b = 1$.
- This contradicts to our assumption that $b < 1$.
- Thus, this case is impossible.

18. Proof, Part 3.2

- We have just proven that we cannot have $a \le t(a, b)$.
- So, we must have $t(a, b) < a$.
- In this case:

$$\max(t(\mu_1(1), t(\mu_2(1), \mu_3(1))), t(\mu_1(2), t(\mu_2(2), \mu_3(2)))) = \max(a, t(a, b)) = a.$$

- Thus, we get

$$(\mu_1 \cap (\mu_2 \cap \mu_3))(1) = \frac{a}{a} = 1, \quad (\mu_1 \cap (\mu_2 \cap \mu_3))(2) = \frac{t(a, b)}{a}.$$

- By associativity, these values should be equal to the values

$$(\mu_1 \cap \mu_2) \cap \mu_3.$$

- By comparing the values of these two fuzzy sets for $x = 2$, we conclude that $\dfrac{t(a, b)}{a} = b$.
- Hence $t(a, b) = a \cdot b$.
- For the values $a > 0$, the proposition is proven.

19. Proof, Part 4

- Let us now consider the case when $a = 0$.
- In this case, by the formulas from Part 3.1 of this proof, the normalized intersection $\mu_1 \cap \mu_2$ is undefined.
- Thus, the set $(\mu_1 \cap \mu_2) \cap \mu_3$ is undefined as well.
- Thus, by associativity, the set $\mu_1 \cap (\mu_2 \cap \mu_3)$ should also be undefined.
- According to the formulas from Part 3.2 of this proof, we get

$$(\mu_2 \cap \mu_3)(1) = 1, \quad (\mu_2 \cap \mu_3)(2) = t(0, b).$$

- Now, from the definition of the "and"-like operation, we get

$$t(\mu_1(1), t(\mu_2(1), \mu_3(1))) = t(0, 1),$$

$$t(\mu_1(2), t(\mu_2(2), \mu_3(2))) = t(1, t(0, b)) = t(0, b).$$

20. Proof, Part 4 (cont-d)

- The fact that the fuzzy set $\mu_1 \cap (\mu_2 \cap \mu_3)$ is undefined means that the corresponding denominator is equal to 0:

$$\max(t(\mu_1(1), t(\mu_2(1), \mu_3(1))), t(\mu_1(2), t(\mu_2(2), \mu_3(2))) =$$

$$\max(t(0,1), t(0,b)) = 0.$$

- Thus, $0 \le t(0,b) \le \max(t(0,1), t(0,b)) = 0$, so indeed $t(0,b) = 0 \cdot b$.
- So, the proposition is also proven for the remaining case $a = 0$.

21. Comment

- In our proof, the only assumption that we made about the operation $t(a, b)$ is that $t(a, 1) = t(1, a) = a$ for all a.
- If we additionally assumed that $t(a, b)$ is a t-norm, then the proof would be much shorter.
- Indeed:
  - for a t-norm we always have $t(a, b) \le a$, so there is no need to consider the case $t(a, b) > a$, and
  - we always have $t(0, b) = 0$, so there is no need to prove this formula.

22. Acknowledgments

This work was supported in part by:

- National Science Foundation grants 1623190, HRD-1834620, HRD-2034030, and EAR-2225395;
- AT&T Fellowship in Information Technology;
- program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and
- a grant from the Hungarian National Research, Development and Innovation Office (NRDI).