We Can Always Reduce a Non-Linear Dynamical System to Linear – at Least Locally – But Does It Help?

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1. Dynamical systems are ubiquitous

- One of the main objectives of science is to predict the future state of different systems.
- The state of a system can be described by the values $x_1, \ldots, x_n$ of different quantities that characterize this system.
- In many cases, the current state of the system uniquely determines the rate $\dot{x}_i$ with which each of the values changes:

$$\dot{x}_i = f_i(x_1, \ldots, x_n), \quad 1 \le i \le n.$$

- Such a system of differential equations is known as a *dynamical system*.
- For a dynamical system, the current state of the system uniquely determines its state at any future (or past) moment of time.

2. In many cases, we have linear dynamical systems

- In many practical situations, the changes are relatively small.
- In such cases, for all moments of time, the values x_i are close to their initial values $x_i^{(0)}$.
- In such cases, it is convenient to describe the state of the system by the differences $y_i \stackrel{\text{def}}{=} x_i - x_i^{(0)}$.
- For these differences, $x_i = x_i^{(0)} + y_i$ and, thus, $\dot{x}_i = \dot{y}_i$.
- In terms of these differences, the equations take the form

$$\dot{y}_i = f\left(x_1^{(0)} + y_1, \ldots, x_n^{(0)} + y_n\right).$$

- Since the values y_i are relatively small, we can safely ignore terms that are quadratic or of higher order in terms of y_i.
- For example, if $y_i \approx 10\%$, then $y_i^2 \approx 1\%$, which is much smaller than y_i.

3. In many cases, we have linear dynamical systems (cont-d)

- Thus, we can expand the right-hand side of the formula in Taylor series in terms of y_i and keep only linear terms in this expansion.
- In this case, the right-hand side of the system becomes linear in $y_1, \ldots, y_n$; $\dot{y}_i = a_i + \sum\limits_{j=1}^{n} a_{i,j} \cdot y_j$ for some a_i and $a_{i,j}$.
- Such dynamical systems are called *linear*.

4. Linear systems are easy to solve

- There are known formulas for solving linear systems.
- Namely, in the absence of the constant terms a_i, the general solution to this system is a linear combination of the terms $t^k \cdot \exp(\lambda \cdot t)$, where:
  - λ is an eigenvalue of the matrix $a_{i,j}$, and
  - k is a non-negative integer which is smaller than the multiplicity of this eigenvalue.
- A simple modification of this formula – in most cases, by adding constants – takes care of the general case.

5. Sometimes, we can reduce a non-linear system to a linear one

- In some cases, the differences y_i are not small.
- So we have to consider the original nonlinear dynamical system.
- For such systems, there is no general way to solve it other than to solve it numerically.
- However, in some cases, it is possible to reduce a nonlinear system to a linear one and thus, to get an explicit solution.
- A known example – for which such a reduction is possible – is the following system: $\dot{x}_1 = a \cdot x_1$, $\dot{x}_2 = b \cdot (x_2 - x_1^2)$.
- In this system, the first equation is linear, but the second one is nonlinear.
- This system can be reduced to a system of linear equations if we add the third variable $x_3 = x_1^2$.
- In this case, the second equation becomes linear: $\dot{x}_2 = b \cdot x_2 - b \cdot x_3$.

6. Sometimes, we can reduce a non-linear system to a linear one (cont-d)

- For the derivative of x_3, we have $\dot{x}_3 = \dfrac{d}{dt}(x_1^2) = 2 \cdot x_1 \cdot \dot{x}_1$.
- Substituting the above expression for $\dot{x}_1$ into this formula, we get

$$\dot{x}_3 = 2 \cdot x_1 \cdot (b \cdot x_1) = 2b \cdot x_1^2.$$

- By definition of x_3 as x_1^2, this implies that for the rate of change of the third variables, we also have a linear equation $\dot{x}_3 = 2b \cdot x_3$.

7. For other nonlinear equations, such a reduction is not known

- Let us consider the simplest possible nonlinear dynamical system.
- The more variables, the more complex the system.
- So the simplest system should have a single variable x_1 – and thus, the dynamical system consists of a single differential equation $\dot{x}_1 = f_1(x_1)$.
- To find the simplest of such systems, we need to select the simplest possible nonlinear function $f_1(x_1)$.
- The simplest possible functions are linear, and the simplest nonlinear functions are quadratic.
- The simplest-to-compute quadratic function is simply the expression x_1^2 that can be computed by a single multiplication.
- So, the simplest possible nonlinear system is the equation $\dot{x}_1 = x_1^2$.
- For this system – as well as for other differential equations with quadratic right-hand sides – no reduction to linear systems is known.

8. Natural questions

- For some nonlinear systems:
  - there is a known reduction to linear ones, and
  - this reduction helps solve the system.
- This fact leads to the following two natural questions:
  - which nonlinear systems can be reduced to linear ones? and
  - if such a reduction is possible, can it help solve the original nonlinear system?
- In this paper, we provide answers to both questions.
- Namely, we show:
  - that *every* nonlinear system can be reduced to a linear one – at least locally, but
  - that this does not help us solve the original nonlinear system – finding this reduction is as complicated as solving the system.

9. Notation

- As we have mentioned, in general:
  - once we know the equations (1) and we know the state $x = (x_1, \ldots, x_n)$ of the system at some moment t_0,
  - we can uniquely determine its state at any other moment of time
  - at least locally, i.e., for times t sufficiently close to t_0.
- The system does not explicitly include time.
- Thus, the transition between the state at moment t_0 and the state at moment t depends only on the difference $t - t_0$.
- Let us denote the state at moment t corresponding to the state x at moment t_0 by $T_{t-t_0}(x)$.
- The components of the state $T_{t-t_0}(x)$ will be denoted by $T_{t-t_0,1}(x), \ldots, T_{t-t_0,n}(x)$.
- In these terms, the state $x = (x_1, \ldots, x_n)$ is transformed into the state

$$(z_1, \ldots, z_n) = (T_{t-t_0,1}(x_1, \ldots, x_n), \ldots, T_{t-t_0,n}(x_1, \ldots, x_n)).$$

10. Notation (cont-d)

- Clearly, if we let the system evolve for time t and then again for time t', this is equivalent to letting it evolve for time $t + t'$.
- So, we have $T_{t+t'}(x) = T_{t'}(T_t(x))$.

11. How to reduce a nonlinear system to linear: idea

- Let us consider the cases when the values of the last variable x_n are close to some number $x_n^{(0)}$.
- In general, unless the system is degenerate and the value x_n does not change at all, the value x_n changes with time.
- For each tuple $(x_1, \ldots, x_{n-1}, x_n)$, we can look for the time τ at which the value x_n is equal to $x_n^{(0)}$, i.e., at which

$$T_{\tau,n}(x_1, \ldots, x_n) = x_n^{(0)}.$$

- To find the unknown time τ, we have one equation with one unknown τ.
- In general, if the number of equations is equal to the number of unknowns, the system has a unique solution – at least locally.
- Thus, we can always find such time τ.

12. Towards the resulting reduction

- Now, instead of the original variables $x_1, \ldots, x_n$, we can use the new variables $(y_1, \ldots, y_n)$ in which:
  - the value y_n is equal to the time τ determined by the formula, and
  - the values $y_1, \ldots, y_i, \ldots, y_{n-1}$ are equal to

$$y_i = T_{\tau,i}(x_1, \ldots, x_n).$$

- How will the variables y_i change with time?
- Due to above formulas, we have $\left(y_1, \ldots, y_{n-1}, x_n^{(0)}\right) = T_\tau(x_1, \ldots, x_n)$.
- Since $\tau = y_n$, we have $\left(y_1, \ldots, y_{n-1}, x_n^{(0)}\right) = T_{y_n}(x_1, \ldots, x_n)$.

13. Towards the resulting reduction (cont-d)

- Thus:
  - to describe x in terms of y,
  - it is sufficient to trace the changes of the state $\left(y_1, \ldots, y_{n-1}, x_n^{(0)}\right)$ back in time for the period $\tau = y_n$:

$$(x_1, \ldots, x_n) = T_{-y_n}\left(y_1, \ldots, y_{n-1}, x_n^{(0)}\right).$$

- In time t, the state x turns into $T_t(x)$.
- By applying the transformation T_t to both sides of the formula, we get $T_t(x_1, \ldots, x_n) = T_{-(y_n - t)}\left(y_1, \ldots, y_{n-1}, x_n^{(0)}\right)$.
- Thus, by definition of the y-state, the y-state corresponding to $T_t(x)$ has the form $(y_1, \ldots, y_{n-1}, y_n - t)$.

14. Towards the resulting reduction (cont-d)

- Thus, with time:
  - the new variables $y_1, \ldots, y_{n-1}$ do not change at all, and
  - the variable y_n linearly decreases with time.
- So, we indeed have the desired reduction; let us summarize it.

15. Resulting reduction

- For each state $x = (x_1, \ldots, x_n)$, let us define the new variables $y = (y_1, \ldots, y_n)$ as follows:
  - y_n is the solution to the equation $T_{y_n,n}(x_1, \ldots, x_n) = x_n^{(0)}$,
  - and for all $i < n$, we have $y_i = T_{y_n,i}(x_1, \ldots, x_n)$.
- For the new variables, the dynamical system has the following form:

$$\dot{y}_1 = \ldots = \dot{y}_{n-1} = 0, \quad \dot{y}_n = -1.$$

- From the new variables y_i, we can get back to the original variables x_i by using the above formula.

16. Example: derivation

- Let us illustrate our reduction on the example of the simplest nonlinear dynamical systems $\dot{x}_1 = x_1^2$.
- This system is easy to solve.
- Indeed, we start with the original equation: $\dfrac{dx_1}{dt} = x_1^2$.
- We then separate the variables by multiplying both sides by dt and dividing both sides by x_1^2.
- This results in: $\dfrac{dx_1}{x_1^2} = dt$.
- If we integrate both sides of this equation, we get $-displaystyle\frac{1}{x_1} + C = t$, where C denotes the integration constant.
- Thus, $\dfrac{1}{x_1} = C - t$, and so, $x_1(t) = \dfrac{1}{C - t}$.
- Let us describe the corresponding function $T_t(x_1)$ that describes the transition from the state at moment 0 to the state at moment t.

17. Example: derivation (cont-d)

- At moment 0, the above formula leads to $x_1(0) = \dfrac{1}{C}$, so $C = \dfrac{1}{x_1(0)}$.
- Substituting this value C into the formula for $x(t)$, we conclude that

$$x_1(t) = \frac{1}{\dfrac{1}{x_1(0)} - t} = \frac{x_1(0)}{1 - x_1(0) \cdot t}.$$

- Thus, $T_t(x_1) = \dfrac{x_1}{1 - x_1 \cdot t}$.
- Let us take $x_1^{(0)} = 1$.
- Then, the value y_1 should be determined by the equation which, in this case, has the form $\dfrac{x_1}{1 - x_1 \cdot y_1} = 1$.
- Hence $x_1 = 1 - x_1 \cdot y_1$, so $x_1 \cdot y_1 = 1 - x_1$, and

$$y_1 = \frac{1 - x_1}{x_1} = \frac{1}{x_1} - 1.$$

18. Example: derivation (cont-d)

- For the new variable y_1, the dynamical system takes a linear form: $\dot{y}_1 = -1$.
- Once we know new state y_1, we can reconstruct the original state x_1 by using the above formula.
- In this case, it takes the form

$$x_1 = T_{-y_1}(1) = \frac{1}{1 - 1 \cdot (-y_1)} = \frac{1}{1 + y_1}.$$

19. Example: summary

- For the nonlinear system $\dot{x}_1 = x_1^2$, we can take the new variable $y_1 = \frac{1}{x_1} - 1$ for which $x_1 = \frac{1}{1 + y_1}$.
- For this new variable, the dynamical system has the form $\dot{y} = -1$.
- Does this reduction help solve the original nonlinear system?
- Not really, since the reduction uses the solution $T_t(x)$ of the system.

20. But can we have a reduction that does help solve the system?

- Not really.
- If we know the solution, then, as we have shown, we can easily find the appropriate reduction.
- Vice versa:
  - if we know a reduction to a linear system,
  - then, since solutions to linear systems are known, we can easily find the solution to the original nonlinear system.
- Because of this:
  - the complexity of finding a reduction to a linear system is exactly the same as
  - the complexity of solving the original nonlinear system.

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