How to Check Continuity Based on Approximate Measurement Results

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1. General setting

- In many practical situations, we know that some physical quantity is uniquely determined by others:
  - the value y of a physical field at a given location is uniquely determined by this location x,
  - the intensity y of an image at some pixel x is uniquely determined by this pixel, etc.
- In mathematical terms, this means that y is a function of x: $y = f(x)$.
- Theoretically, there are infinitely many possible locations x, and, correspondingly, infinitely many corresponding values $y = f(x)$.

2. General setting (cont-d)

- In practice, however, during a finite period of time, we can only measure finitely many values.
- Thus, at any moment of time:
  - after all the measurements,
  - we know finitely many values $\widetilde{y}_1, \ldots, \widetilde{y}_n$ measured at corresponding locations $x_1, \ldots, x_n$.

3. Need to check continuity

- In many situations, a natural conjecture is that:
  - the dependence $y = f(x)$ is continuous,
  - so that small changes in x lead to small changes in y.
- This conjecture usually involves:
  - some upper bounds d_{ij}
  - on the difference between the values y_i and y_j of the quantity y at two different points x_i and x_j:

$$|y_i - y_j| \le d_{ij}.$$

- How can we check this conjecture?

4. The condition is straightforward, but checking this condition is not always straightforward

- If we knew the exact values $y_1, \ldots, y_n$, then checking would be straightforward.
- We could simply check whether these values satisfy all the desired inequalities.
- In reality, of course, measurements are never absolutely accurate.
- The measurement result $\widetilde{y}_i$ is, in general, somewhat different from the actual value y_i.
- In many cases:
  - the only information that we have about each measurement error $\Delta y_i \stackrel{\text{def}}{=} \widetilde{y}_i - y_i$ is
  - the upper bound Δ_i on its absolute value:

$$|\Delta y_i| = |\widetilde{y}_i - y_i| \le \Delta_i.$$

5. The condition is straightforward, but checking this condition is not always straightforward (cont-d)

- How can we check continuity based on the approximate measurement results $\widetilde{y}_i$?
- We can use the fact that $|a+b| \le |a| + |b|$.
- We thus derive, from the above inequalities, the following inequality containing only the measurement results:

$$|\widetilde{y}_i - \widetilde{y}_j| \le |\widetilde{y}_i - y_i| + |y_i - y_j| + |y_j - \widetilde{y}_j| \le \Delta_i + d_{ij} + \Delta_j, \text{ i.e.,}$$

$$|\widetilde{y}_i - \widetilde{y}_j| \le \Delta_i + d_{ij} + \Delta_j.$$

- So, if some measurement results do not satisfy this inequality, then clearly the dependence is not continuous.
- What if all the derived inequalities are true?

6. The condition is straightforward, but checking this condition is not always straightforward (cont-d)

- Does this imply that the continuity hypothesis can still be true for some possible values y_i?
- In this talk, we prove that the answer to this question is positive:
  - if all the derived inequalities are satisfied,
  - then there exist values y_i that are Δ_i-close to the measurement values and that satisfy the continuity inequality.

7. Enter triangle inequality

- Let us first prove that in the continuity inequality, we can always:
  - replace the original values d_{ij}
  - with the values d'_{ij} that satisfy the triangle inequality

$$d'_{ik} \le d'_{ij} + d'_{jk}.$$

- *For every set of values $d_{ij} = d_{ji} \ge 0$, there exist values d'_{ij}:*
  - *that satisfy the triangle inequality and*
  - *for which, for all possible values $y_1, \ldots, y_n$, the following two conditions are equivalent to each other:*
    - *for all i and j, we have $|y_i - y_j| \le d_{ij}$, and*
    - *for all i and j, we have $|y_i - y_j| \le d'_{ij}$.*
- **Proof.** Let us show that the equivalence is satisfied for the following values: $d'_{ij} \stackrel{\text{def}}{=} \min\{d_{ii_1} + d_{i_1 i_2} + \ldots + d_{i_k j}\}$.
- Here minimum is taken over all possible sequence $i_1, \ldots, i_k$, for all $k \ge 0$.

8. Triangle inequality (cont-d)

- Let us first prove that the values d'_{ij} satisfy the triangle inequality

$$d'_{ik} \leq d'_{ij} + d'_{jk}.$$

- Indeed, d'_{ij} is equal to one of the sums

$$d_{ii_1} + d_{i_1 i_2} + \ldots + d_{i_k j}.$$

- Namely, d'_{ij} is equal to the smallest of these sums.
- Similarly, d'_{jk} is equal to the smallest of the sums

$$d_{jj_1} + d_{j_1 j_2} + \ldots + d_{j_\ell j}.$$

- Thus, the sum $d'_{ij} + d'_{jk}$ is equal to the sum of these two sums:

$$d'_{ij} + d'_{jk} = d_{ii_1} + d_{i_1 i_2} + \ldots + d_{i_k j} + d_{jj_1} + d_{j_1 j_2} + \ldots + d_{j_\ell j}.$$

- By definition, d'_{ik} is equal to the smallest of such sums, and is, thus, smaller than or equal to the above sum.

9. Triangle inequality (cont-d)

- The triangle inequality is proven.
- Let us prove that if the continuity condition is satisfied for d_{ij}, then it is also satisfied for d'_{ij}.
- Indeed, for each sequence $i_1, \ldots, i_k$, we have

$$|y_i - y_j| \le |y_i - y_{i_1}| + |y_{i_1} - y_{i_2}| + \ldots + |y_{i_k} - y_j|.$$

- Due to the continuity condition for d_{ij}, we have

$$|y_i - y_{i_1}| \le d_{ii_1}, \quad |y_{i_1} - y_{i_2}| \le d_{i_1 i_2}, \quad \ldots, \quad |y_{i_k} - y_j| \le d_{i_k j}.$$

- Thus, the previous inequality implies that

$$|y_i - y_j| \le d_{ii_1} + d_{i_1 i_2} + \ldots + d_{i_k j}.$$

- So, the difference $|y_i - y_j|$ is smaller or equal than each sum

$$d_{ii_1} + d_{i_1 i_2} + \ldots + d_{i_k j}.$$

10. Triangle inequality (cont-d)

- Thus, the difference $|y_i - y_j|$ is smaller than the smallest of these sum, i.e., smaller than or equal to d'_{ij}.
- This is exactly continuity condition for d'_{ij}.
- To complete the proof, it is now sufficient to prove that d'_{ij}-continuity implies d_{ij}-continuity.
- Indeed, by definition, d'_{ij} is the smallest of the sums

$$d_{ii_1} + d_{i_1 i_2} + \ldots + d_{i_k j}.$$

- It is, thus, smaller than or equal to each of these sums.
- In particular, for $k = 0$, we conclude that $d'_{ij} \le d_{ij}$.
- Thus, the inequality $|y_i - y_j| \le d'_{ij}$ implies that $|y_i - y_j| \le d_{ij}$.
- So, indeed, d'_{ij}-continuity implies d_{ij}-continuity.
- The proposition is proven.

11. Triangle inequality (cont-d)

- Thus, without losing generality, we can assume that the values d_{ij} satisfy the triangle inequality.
- Now, we are ready to formulate our main result.

12. Proposition

- *Let n be an integer, and let us assume that we have values $\widetilde{y}_1, \ldots, \widetilde{y}_n$, $\Delta_1, \ldots, \Delta_n$, and $d_{ij} = d_{ji} \geq 0$ that satisfy the triangle inequality.*
- *Then, the following two conditions are equivalent to each other:*
  - *the given values $\widetilde{y}_i$ satisfy the derived inequality for all i and j, and*
  - *there exist values $y_1, \ldots, y_n$ which are close to $\widetilde{y}_i$ and which are d_{ij}-continuous.*

13. Proof

- We will prove the equivalence by induction.
- Namely, the second condition has the following form:

$$\exists y_1\, \exists y_2\, \ldots \exists y_n \text{ (closeness and continuity conditions are satisfied).}$$

- We will show that we can:
  - "eliminate" quantifiers one by one, starting with the variable y_n,
  - i.e., replace each statement with an equivalent statement without the corresponding variable.
- We will also prove that after eliminating all the variables y_i, we will get exactly the derived inequalities.
- We start with the system of inequalities describing closeness and continuity.
- We have shown that these inequalities imply the derived inequalities.

14. Proof (cont-d)

- So, we can add derived inequalities to this system – and this will be equivalent to the original system.
- Similarly, from $|y_i - y_j| \le d_{ij}$ and $|y_j - \widetilde{y}_j| \le \Delta_j$, we can conclude that
$$|y_i - \widetilde{y}_j| \le |y_i - y_j| + |y_j - \widetilde{y}_j| \le \Delta_j + d_{ij}.$$
- Thus, we get the following equivalent form of the second condition:
$$y_j - d_{ij} \le y_i \le y_j + d_{ij};$$
$$\widetilde{y}_i - \Delta_i \le y_i \le \widetilde{y}_i + \Delta_i;$$
$$\widetilde{y}_j - \Delta_i - \Delta_j - d_{ij} \le \widetilde{y}_i \le \widetilde{y}_j + \Delta_i + \Delta_j + d_{ij};$$
$$\widetilde{y}_j - \Delta_j - d_{ij} \le y_i \le \widetilde{y}_j + \Delta_j + d_{ij}.$$

15. Proof (cont-d)

- Let us show, by inverse induction, that for each $a = n, n-1, \ldots, 0$:
  - the derived inequalities are equivalent to
  - the existence of the variables $y_1, \ldots, y_a$ for which the above 5 inequalities are satisfied.
- Indeed, let us prove that we can eliminate inequalities including y_a while keeping the condition equivalent.
- To prove this, let us consider all inequalities containing this variable y_a:
  1. $y_j - d_{aj} \le y_a \le y_j + d_{aj}$;
  2. $\widetilde{y}_a - \Delta_a \le y_a \le \widetilde{y}_a + \Delta_a$;
  3. $\widetilde{y}_j - \Delta_j - d_{aj} \le y_a \le \widetilde{y}_j + \Delta_j + d_{aj}$.
- Each of these inequalities provides lower and upper bounds for y_a.
- Thus, for such y_a to exist, every lower bound must be smaller or equal to any upper bound.

16. Proof (cont-d)

- Vice versa, if every lower bound is smaller than or equal to every upper bound, then the desired value y_a exists.
- E.g., we can take, as y_a, the largest of the lower bounds.
- So let us show that indeed, in the above system of 3 inequalities, each lower bound is smaller than or equal that every upper bound.
- Two bounds from the same inequality clearly satisfy this property.
- So, to prove this fact, we need to compare:
  - each lower bound from a double inequality
  - with upper bounds from other double inequalities.
- Let us consider all possible cases.

17. Proof (cont-d)

- Comparing lower bound from one of the type-1 inequalities with an upper bound from another type-1 inequality, we get:

$$y_j - d_{aj} \le y_{j'} + d_{aj'}, \text{ i.e., equivalently:}$$

$$y_j \le y_{j'} + d_{aj} + d_{aj'}.$$

- This is true, since continuity implies that $y_j \le y_{j'} + d_{jj'}$, and, due to the triangle inequality, we have $d_{jj'} \le d_{aj} + d_{aj'}$.
- So $y_{j'} + d_{jj'} \le y_{j'} + d_{aj} + d_{aj'}$ and thus, by transitivity,

$$y_j \le y_{j'} + d_{aj} + d_{aj'}.$$

18. Proof (cont-d)

- Comparing lower bound from one of the type-1 inequalities with a type-2 upper bound, we get:

$$y_j - d_{aj} \le \widetilde{y}_a + \Delta_a, \text{ i.e., equivalently,}$$

$$y_j \le \widetilde{y}_a + \Delta_a = d_{aj}.$$

- This inequality is true: it is a particular case of the type-3 inequalities.

19. Proof (cont-d)

- Comparing type-1 lower bound with a type-2 upper bound, we get:

$$y_j - d_{aj} \le \widetilde{y}_{j'} + \Delta_{j'} + d_{aj'}, \text{ i.e., equivalently:}$$

$$y_j \le \widetilde{y}_{j'} + \Delta_a + d_{aj} + d_{aj'}.$$

- One of type-3 inequalities has the form

$$y_j \le \widetilde{y}_{j'} + \Delta_a + d_{jj'}.$$

- Due to triangle inequality $d_{jj'} \le d_{aj} + d_{aj'}$, so we indeed get the desired inequality.

20. Proof (cont-d)

- Comparing type-2 lower bound with a type-3 upper bound, we get:

$$\widetilde{y}_a - \Delta_a \le \widetilde{y}_j + \Delta_j + d_{aj}, \text{ i.e., equivalently,}$$

$$\widetilde{y}_a \le \widetilde{y}_j + \Delta_a + \Delta_j + d_{aj}.$$

- This is true, since it is one of 5 inequalities.

21. Proof (cont-d)

- Finally, comparing the type-3 lower bound with another type-3 upper bound from, we get:

$$\widetilde{y}_j - \Delta_j - d_{aj} \le \widetilde{y}_{j'} + \Delta_{j'} + d_{aj'}, \text{ i.e., equivalently:}$$

$$\widetilde{y}_j \le \widetilde{y}_{j'} + \Delta_j + \Delta_{j'} + d_{aj} + d_{aj'}.$$

- One of the 5 inequalities has the form:

$$\widetilde{y}_j \le \widetilde{y}_{j'} + \Delta_j + \Delta_{j'} + d_{jj'}.$$

- Due to triangle inequality $d_{jj'} \le d_{aj} + d_{aj'}$, so we indeed get the desired inequality.

- The proposition is proven.

22. Acknowledgments

This work was supported in part by:

- National Science Foundation grants 1623190, HRD-1834620, HRD-2034030, and EAR-2225395;
- AT&T Fellowship in Information Technology;
- a grant from the Hungarian National Research, Development and Innovation Office (NRDI), and
- by the Institute for Risk and Reliability, Leibniz University Hannover, Germany.