What If the Resulting Interval Is Too Wide: From a Heuristic Fuzzy-Technique Idea to a Mathematically Justified Approach

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1.   Practical problem

- When we design an engineering structure, we need to make sure:
    - that several important characteristics – such as the stress level
    - do not exceed a given threshold y_0.

- Usually, we can use known equations of mechanics. They enable us to compute the value y if we know the exact values of all the design parameters $x_1, \ldots, x_n$, such as:
    - the sizes of the beams and other components,
    - the mechanical characteristics of the material at different locations, etc.

- In other words, we have an algorithm $y = f(x_1, \ldots, x_n)$.

- So, a natural idea is to select these parameters in such a way that the corresponding threshold y_0 is not exceeded, i.e., that

$$f(x_1, \ldots, x_n) \leq y_0.$$

2. Need to take uncertainty into account

- In practice, we can only maintain the design parameters with some accuracy.

- For example, even when we mass produce beams, the beam's Young modulus slightly changes from one sample to the other.

- Similarly, the beam's length is only maintained with some given tolerance.

- Because of this, it is not sufficient to guarantee the desired inequality $f(x_1, \ldots, x_n) \leq y_0$ for the nominal values of the parameters x_i.

- We also need to make sure that this inequality is satisfied:

  - for the actual values of these parameters,
  - which are, in general, different from the nominal values.

3.   Ideal case, when we know all the probabilities

- In the ideal case, we know the probability of all possible tuples

$$(x_1, \ldots, x_n).$$

- In other words, we know the probability distribution on the set of all these tuples.

- Individual quantities may be independent, they may be correlated.

- Based on this known distribution, we can find the probability of different values y.

- Suppose that the probability of y not exceeding y_0 is sufficiently small.

- In other words, smaller than or equal to some small threshold probability p_0.

- Then the design is acceptable.

- Otherwise the design is not acceptable.

4. Realistic case: we do not know the probabilities

- In practice, usually, we do not know the probabilities.

- Moreover, there may not be any probabilities.

- Indeed, the frequency of different values may change with each – even minor – change of the manufacturing process of even of the input materials.

- In such cases, the only thing we know are tolerance intervals $[\underline{x}_i, \overline{x}_i]$ that are guaranteed to contain all possible values of x_i.

- For some quantities, there may not be a prescribed tolerance interval.

- For such quantities, we can take:

  - as $\underline{x}_i$, the smallest of the measured values x_i, and
  - as $\overline{x}_i$, the largest of the measured values x_i.

- As a result, for each quantity x_i, we have an interval $[\underline{x}_i, \overline{x}_i]$ that contains the actual value x_i.

5. We can apply interval computations

- All we know about each quantity x_i is the interval of its possible values.

- So, a natural idea is:
  - to consider all possible tuples consisting of such x_i, and
  - to compute the corresponding range of the function $f(x_1, \ldots, x_n)$:

  $$[\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

- This range is usually denoted by $f([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])$.

- The problem of computing the range of a function over given intervals is one of the main problems of *interval computations.*

- In interval computations, many methods have been developed to compute this range.

- We can thus use one of these methods and compute the bounds $\underline{y}$ and $\overline{y}$.

6. Examples

- Let us list two examples that will be useful in our analysis.

- In both examples, we assume that we know x_1 exactly, i.e.,

$$[\underline{x}_1, \overline{x}_1] = \{x_1\}.$$

- In the first example, $f(x_1, x_2) = x_1 \cdot x_2$.

- Then $x_1 \cdot [\underline{x}_2, \overline{x}_2]$ means the range of all the values $x_1 \cdot x_2$ when $x_2 \in [\underline{x}_2, \overline{x}_2]$.

- When $x_1 \geq 0$, then the value $x_1 \cdot x_2$ increases with x_2 and thus, the desired range is equal to $[x_1 \cdot \underline{x}_1, x_1 \cdot \overline{x}_2]$.

- When $x_1 \leq 0$, then the value $x_1 \cdot x_2$ decreases with x_2 and thus, the desired range is equal to $[x_1 \cdot \overline{x}_1, x_1 \cdot \underline{x}_2]$.

- In particular, for $x_1 = -1$, we get $(-1) \cdot [\underline{x}_2, \overline{x}_2] = [-\overline{x}_2, -\underline{x}_2]$.

- The value $(-1) \cdot x$ is usually denoted by $-x$, so it is reasonable to similarly denote this range by $-[\underline{x}_1, \overline{x}_2]$.

7.   Examples (cont-d)

- In the second example, instead of multiplication, we have addition $f(x_1, x_2) = x_1 + x_2$.

- Then, since $x_1 + x_2$ is an increasing function of x_2, we have

$$x_1 + [\underline{x}_2, \overline{x}_2] = [x_1 + \underline{x}_2, x_1 + \overline{x}_2].$$

8. Proposition

- *For every two values $\underline{y} \leq \overline{y}$ and for each y_0, the following two conditions are equivalent to each other:*

  - *we have $y \leq y_0$ for all y from the interval $[\underline{y}, \overline{y}]$, and*
  - *we have $\overline{y} \leq y_0$.*

- **Proof.** If $\overline{y} \leq y_0$, then we can guarantee that $y \leq y_0$ for all $y \leq \overline{y}$ (so the design is acceptable).

- Otherwise, if $\overline{y} > y_0$, that would mean that

  - for some possible value y – namely, for the value $y = \overline{y}$,
  - the corresponding value y exceeds the desired threshold.

- Thus, the design is not acceptable).

- The proposition is proven.

9. The resulting interval $[\underline{y}, \overline{y}]$ is often too wide: a problem

- This approach has a problem.

- The resulting interval $[\underline{y}, \overline{y}]$ is often too wide.

- E.g., the upper endpoint $\overline{y}$ is much larger than all observed values y.

- As a result, to satisfy the criterion $\overline{y} \leq y_0$, we need to impose too strong restrictions on the design.

10. Why is the interval too wide?

- The reason for the above discrepancy is that:
  - the worst-case scenario $y = \overline{y}$ usually occurs
  - when each quantity x_i attains one of its extreme values $\underline{x}_i$ or $\overline{x}_i$.

- This is definitely true for linear functions

$$f(x_1, \ldots, x_n) = a_0 + a_1 \cdot x_1 + \ldots + a_n \cdot x_n.$$

- For them, the maximum with respect to each variable is always attained at one of the endpoints of its interval.

- This is, thus:
  - true for all smooth functions $f(x_1, \ldots, x_n)$
  - which, in the small vicinity of each tuple, can be reasonably accurately approximated by a linear function.

11. Why is the interval too wide (cont-d)

- For each quantity x_i, it is possible – although not very probable – to attain one of the extreme values.

- This is how we selected these extreme values.

- However, from the commonsense viewpoint, it is not very probable that all the quantities attain their extreme values.

- For example, in a group of people:
  - we can have the tallest person and the shortest person,
  - we can have the smartest person and the least smart person,
  - we can have the oldest person and the youngest person, etc.

- However, in a randomly selected group of people, it is highly improbable that:
  - the tallest person from this group is also
  - either the smartest or the least smart *and* either the oldest or the youngest, etc.

12. Why is the interval too wide (cont-d)

- So, a natural problem is how to take this into account?

- How to come up with a procedure that would return a narrower – and thus, more practical – interval of possible values of y.

13.   What we do in this talk

- The problem that we need to solve is not formulated in precise mathematical terms.

- Our explanation for this problem uses imprecise commonsense statements.

- These statements are formulated by using imprecise words from natural language.

- We need to describe this problem in precise terms.

- It is therefore reasonable to use techniques specifically designed:
  - to transform such imprecise ("fuzzy") natural-language statements
  - into precise mathematical terms.

- There are the technique of *fuzzy logic*.

- From the commonsense viewpoint, the solution based on fuzzy logic makes perfect sense.

14.   What we do in this talk (cont-d)

- However:
  - from the engineering viewpoint,
  - using heuristic methods like fuzzy logic is not good enough to use it in serious applications.

- For serious practical applications, we need to have a mathematically justified method.

- Such a method is described too.

- Interestingly, the resulting mathematically justified method leads to the exact same conclusion as the heuristic method.

- If this is the case, why did we need to describe the heuristic method at all?

- The answer is that:
  - while having a purely mathematical justification is good,
  - it is more convincing when we also have a commonsense-based explanation,
  - and this is what fuzzy-based approach does.

16. What is fuzzy: general reminder

- In fuzzy technique, we ask an expert to assign:
  - to each possible event,
  - the degree of confidence – a number from the interval $[0, 1]$ – that this event will happen.

- In our case, events are possible values of the quantity.

- In this case, to each value x, we assign a degree $m(x)$.

- The corresponding function is known as a *membership function*, or, alternatively, as the *fuzzy set*.

- **Definition.** *By a* membership function *(or, alternatively, a* fuzzy set*), we mean a function from some set to the interval* $[0, 1]$.

- Usually, membership functions are *normalized*, in the sense that they describe "relative" degrees, so that the largest of these degrees is 1.

- Intuitively, for close values x, the degrees should be close, so the membership function should be continuous.

17. Which membership function should we choose

- We assume that values $x < \underline{x}$ and $x > \overline{x}$ are not possible.

- Thus, for these values x, we have $m(x) = 0$.

- Since the membership function is continuous, for $x \to \underline{x}$ and $x \to \overline{x}$, we get $m(\underline{x}) = m(\overline{x}) = 0$.

- In other words, having extreme values is not very probable.

- Intuitively, the further away from the endpoints, the more probable is the corresponding value.

- So, the largest degree of confidence should be attained at the point whose smallest distance from endpoints is the largest.

- One can easily show that this is the midpoint $\widetilde{x} \stackrel{\text{def}}{=} \dfrac{\underline{x} + \overline{x}}{2}$ of the interval $[\underline{x}, \overline{x}]$.

18.   Proposition

- *Among all the points from an interval $[\underline{x}, \overline{x}]$, its midpoint has the largest value of the quantity $\min(|x - \underline{x}|, |x - \overline{x}|)$.*

- **Proof.** Indeed, for the midpoint, the maximized quantity is equal to half of the interval's width.

- For any other point, the distance to one of the endpoints is smaller than the half-width.

- Thus, the minimum $\min(|x - \underline{x}|, |x - \overline{x}|)$ is also smaller than the half-width.

- The statement is proven.

19.   Fuzzy (cont-d)

- Since membership functions are normalized, we thus have $m(\widetilde{x}) = 1$.

- So, we have $m(\underline{x}) = 0$, $m(\widetilde{x}) = 1$, and $m(\overline{x}) = 0$.

- The distance from the midpoint to each to the endpoints is equal to half of the interval's width: $\widetilde{x} - \underline{x} = \overline{x} - \widetilde{x} = \dfrac{\overline{x} - \underline{x}}{2}$.

- There are many ways to extend this function to intermediate values:
  - between $\underline{x}$ and $\widetilde{x}$, and
  - between $\widetilde{x}$ and $\overline{x}$.

- A natural idea is to use the fact that for close values x, the values $m(x)$ should be close.

20. Fuzzy (cont-d)

- We want to make sure that this property is maximally satisfied for each positive real number $\varepsilon > 0$:

  - among all the functions $m(x)$ that attains given values at the three points $\underline{x}$, $\widetilde{x}$, and $\overline{x}$,
  - the value $v(\varepsilon) \overset{\text{def}}{=} \max(|m(x) - m(x')| : |x - x'| \le \varepsilon)$ is the smallest possible.

- It turns out that this requirement uniquely determines the corresponding membership function.

- *Let us consider all continuous functions $m : [\underline{x}, \overline{x}] \mapsto [0, 1]$ for which $m(\underline{x}) = m(\overline{x}) = 0$ and $m(\widetilde{x}) = 1$.*

- *Then, the following function $m_0(x)$ attains, for all $\varepsilon > 0$, the smallest possible value of the quantity*

$$v(\varepsilon) = \max(|m(x) - m(x')| : |x - x'| \le \varepsilon) :$$

$$m_0(x) = \frac{x - \underline{x}}{\widetilde{x} - \underline{x}} \text{ for } x < \widetilde{x} \text{ and } m_0(x) = \frac{\overline{x} - x}{\overline{x} - \widetilde{x}} \text{ for } x > \widetilde{x}.$$

21. Fuzzy (cont-d)

- *Comment.* This function $m_0(x)$ linearly increases from 0 to 1, and then linearly decreases from 1 to 0.

- **Proof.** For the piece-wise linear function $m_0(x)$, for each $\varepsilon \leq \widetilde{x} - \underline{x} = \overline{x} - \widetilde{x}$, we have $v(\varepsilon) = \dfrac{\varepsilon}{\widetilde{x} - \underline{x}}$.

- Thus, for the membership function $m_0(x)$ with the smallest $v(\varepsilon)$, we should have $v(\varepsilon) \leq \dfrac{\varepsilon}{\widetilde{x} - \underline{x}}$.

- In particular, for $\varepsilon_0 = \dfrac{\widetilde{x} - \underline{x}}{n}$, we should have $v(\varepsilon_0) \leq 1/n$.

- This means, in particular, that for every integer $i < n$, we must have

$$|m(\underline{x} + (i + 1) \cdot \varepsilon_0) - m(\underline{x} + i \cdot \varepsilon_0)| \leq 1/n.$$

22. Fuzzy (cont-d)

- We cannot have $m(\underline{x} + (i+1) \cdot \varepsilon_0) - m(\underline{x} + i \cdot \varepsilon_0) < 1/n$ for some i, since then we would have

$$1 = m(\widetilde{x}) - m(\underline{x}) = m(\underline{x} + n \cdot \varepsilon_0) - m(\underline{x}) =$$

$$(m(\underline{x} + \varepsilon_0) - m(\underline{x})) + \ldots + (m(\underline{x} + (i+1) \cdot \varepsilon_0) - m(\underline{x} + i \cdot \varepsilon_0)) +$$

$$\ldots + m(\underline{x} + n \cdot \varepsilon_0) - m(\underline{x} + (n-1) \cdot \varepsilon_0) < 1/n + \ldots + 1/n + \ldots + 1/n = 1,$$

$$\text{i.e., } 1 < 1.$$

- The difference $m(\underline{x} + (i+1) \cdot \varepsilon_0) - m(\underline{x} + i \cdot \varepsilon)$ is smaller than or equal to $1/n$.

- This difference cannot be smaller than $1/n$.

- So, it must be equal to $1/n$.

23.   Fuzzy (cont-d)

- Thus, for all i, we have

$$m(\underline{x} + i \cdot \varepsilon_0) = m(\underline{x} + i \cdot \varepsilon_0) - m(\underline{x}) =$$
$$(m(\underline{x} + \varepsilon_0) - m(\underline{x})) + \ldots + (m(\underline{x} + i \cdot \varepsilon_0) - m(\underline{x} + (i - 1) \cdot \varepsilon_0)) =$$
$$1/n + \ldots + 1/n = i/n.$$

- Substituting the expression for ε_0 into this formula, we conclude that $m(\underline{x} + (i/n) \cdot (\widetilde{x} - \underline{x})) = i/n$ for all rational numbers $i/n \in [0, 1]$.

- Since the membership function is continuous, we conclude that

$$m(\underline{x} + r \cdot (\widetilde{x} - \underline{x})) = r \text{ for all real numbers } r \in [0, 1].$$

- So, on the interval $[\underline{x}, \widetilde{x}]$, the membership function with the smallest $v(\varepsilon)$ indeed linearly grows from 0 to 1.

- Similarly, we can prove that on the interval $[\widetilde{x}, \overline{x}]$, the membership function with the smallest $v(\varepsilon)$ indeed linearly decreases from 1 to 0.

- The proposition is proven.

24. How to process fuzzy data: reminder

- Once we selected the membership functions $m_i(x_i)$ for each input x_i, we can then compute:

  - for each value x_i from the interval $[\underline{x}_i, \overline{x}_i]$,

  - the degree of confidence that this value is possible.

- Based on this information, we must decide, for each value y from the interval $[\underline{y}, \overline{y}]$, to what extent this value y is possible.

- The value y is possible if for some tuple $(x_1, \ldots, x_n)$ for which $y = f(x_1, \ldots, x_n)$:

  - the value x_1 is possible, *and*

  - the value x_2 is possible, ...

- In other words, either the above "and"-statement is true for one of the tuples, *or* for another tuples, etc.

25. How to process fuzzy data (cont-d)

- The above description is obtained:
  - from the basic statements of the type "x_i is possible" – for which we know their degrees of confidence
  - by using logical connectives "and" (usually denoted &) and "or" (usually denoted $\vee$).

- So:
  - to find the degree to which y is possible,
  - we must learn to propagate degrees of confidence through these logical connectives.

- In other words:
  - given degrees of confidence a and b in statements A and B,
  - produce the estimate degrees of confidence $f_\&(a, b)$ and $f_\vee(a, b)$ for statements $A \& B$ and $A \vee B$.

26.   How to process fuzzy data (cont-d)

- To find the corresponding functions $f_\&(a,b)$ and $f_\vee(a,b)$, let us use commonsense requirements.

- Let us start with "and".

- First, $A\,\&\,B$ implies both A and B.

- So, our degree of confidence in $A\,\&\,B$ cannot exceed our degrees of confidence in A and in B: $f_\&(a,b) \le a$ and $f_\&(a,b) \le b$.

- Second, $A\,\&\,A$ means the same as A.

- So, it is reasonable to require that the degree of confidence in $A\,\&\,A$ be the same as the degree of confidence in A.

- In other words, we should have $f_\&(a,a) = a$ for all a.

27. How to process fuzzy data (cont-d)

- Third:

  - if our degree of confidence in one of the statements A or B increases,

  - our confidence in $A \, \& \, B$ should also increase – or at least remain the same.

- So, if $a \leq a'$ and $b \leq b'$, then we should have $f_\&(a, b) \leq f_\&(a', b)$ and $f_\&(a, b) \leq f_\&(a, b')$.

- One can show that these three conditions are sufficient to uniquely determine the function $f_\&(a, b)$.

28. Result for "and"

- *We say that a function $f_\& : [0,1] \times [0,1] \to [0,1]$ is an "and"-operation is it satisfies the following properties:*

  - *for all a and b, we have $f_\&(a,b) \leq a$ and $f_\&(a,b) \leq b$;*
  - *for all a, we have $f_\&(a,a) = a$; and*
  - *if $a \leq a'$ and $b \leq b'$, then:*

    $$f_\&(a,b) \leq f_\&(a',b) \text{ and } f_\&(a,b) \leq f_\&(a,b').$$

- **Proposition.** *The only "and"-operation is $f_\&(a,b) = \min(a,b)$.*

- **Proof.** It is easy to prove that the function $\min(a,b)$ is an "and"-operation.

- To complete the proof, we need to show that every "and"-operation coincides with $\min(a,b)$.

- If $a \leq b$, then, due to monotonicity $f_\&(a,a) \leq f_\&(a,b)$, so

$$a \leq f_\&(a,b).$$

29. Result for "and" (cont-d)

- On the other hand, by the first bullet, $f_\&(a, b) \leq a$.

- Thus, $f_\&(a, b) = a$.

- Similarly, if $b \leq a$, then, due to monotonicity, $f_\&(b, b) \leq f_\&(a, b)$, so

$$b \leq f_\&(a, b).$$

- On the other hand, by the first bullet, $f_\&(a, b) \leq b$.

- Thus, $f_\&(a, b) = b$.

- These two cases can be described by a single formula

$$f_\&(a, b) = \min(a, b).$$

- The proposition is proven.

30. Case of "or"

- First, each of the two statements A and B implies $A \vee B$.

- So, our degree of confidence in $A \vee B$ cannot be smaller that our degrees of confidence in A and in B:

$$a \leq f_\vee(a, b) \text{ and } b \leq f_\vee(a, b).$$

- Second, $A \vee A$ means the same as A.

- So, it is reasonable to require that the degree of confidence in $A \vee A$ be the same as the degree of confidence in A.

- So, we should have $f_\vee(a, a) = a$ for all a.

- Third:

  - if our degree of confidence in one of the statements A or B increases,

  - our confidence in $A \vee B$ should also increase – or at least remain the same.

31.   Case of "or" (cont-d)

- So, if $a \leq a'$ and $b \leq b'$, then we should have $f_\vee(a, b) \leq f_\vee(a', b)$ and $f_\&(a, b) \leq f_\vee(a, b')$.

- One can show that these three conditions are sufficient to uniquely determine the function $f_\vee(a, b)$.

32.   Result for "or"

- *We say that a function $f_\vee : [0,1] \times [0,1] \to [0,1]$ is an "or"-operation is it satisfies the following properties:*

  - *for all a and b, we have $a \le f_\vee(a,b)$ and $b \le f_\vee(a,b)$;*
  - *for all a, we have $f_\vee(a,a) = a$; and*
  - *if $a \le a'$ and $b \le b'$, then:*

    $$f_\vee(a,b) \le f_\vee(a',b) \text{ and } f_\vee(a,b) \le f_\vee(a,b').$$

- **Proposition.** *The only "or"-operation is $f_\vee(a,b) = \max(a,b)$.*

- **Proof.** It is easy to prove that the function $\max(a,b)$ is an "or"-operation.

- To complete the proof, we need to show that every "or"-operation coincides with $\max(a,b)$.

- If $a \le b$, then, due to monotonicity $f_\vee(a,b) \le f_\vee(b,b)$, so $f_\vee(a,b) \le b$.

- On the other hand, by the first bullet, $b \le f_\vee(a,b)$.

- Thus, $f_\vee(a, b) = b$.

- Similarly, if $b \le a$, then, due to monotonicity, $f_\vee(a, b) \le f_\vee(a, a)$, so

$$f_\vee(a, b) \le a.$$

- On the other hand, by the first bullet, $a \le f_\vee(a, b)$.

- Thus, $f_\vee(a, b) = a$.

- These two cases can be described by a single formula

$$f_\vee(a, b) = \max(a, b).$$

- The proposition is proven.

34. Resulting membership function for y

- Let us replace "and" with min and "or" with max.

- Then the above description of when y is possible leads to the following formula:

$$m(y) = \max(\min(m_1(x_1), \ldots, m_n(x_n)) : f(x_1, \ldots, x_n) = y).$$

- This formula was first proposed by Lotfi Zadeh, father of fuzzy logic and is, thus known as *Zadeh's extension principle*.

35. How can we compute Zadeh's extension principle

- To describe a membership function $m(x)$, it is sufficient to know:

  - for each $\alpha \in (0, 1]$,
  - the set of all the values x for which $m(x) \geq \alpha$.

- Each such set $\mathbf{x}(\alpha) \stackrel{\text{def}}{=} \{x : m(x) \geq \alpha\}$ is knows as the α-*cut* of the original fuzzy set.

- Indeed, $m(x)$ is the largest value α for which $m(x) \geq \alpha$, i.e., $x \in \mathbf{x}(\alpha)$.

- So, to find the desired membership function $m(y)$, it is sufficient to find the corresponding α-cuts.

- It is known that for each α, the α-cut of the membership function $m(y)$ is equal to the range of $f(x_1, \ldots, x_n)$ on α-cuts for x_i:

$$\mathbf{y}(\alpha) = f(\mathbf{x}_1(\alpha), \ldots, \mathbf{x}_n(\alpha)).$$

- So, we can use known interval computation techniques to compute this range.

36. Towards the resulting recommendation

- Suppose that we have computed, for each possible value y, the degree $m(y)$ with which this value is possible.

- Then, it makes sense:
  - to dismiss the values y for which this degree is close to 0, and
  - to only consider values y for which this degree is sufficiently large, i.e., larger than some threshold α_0.

- Thus:
  - instead of the too-wide range of the function $f(x_1, \ldots, x_n)$ over the original intervals $[\underline{x}_i, \overline{x}_i]$,
  - it makes sense to consider its range over the α-cuts $\mathbf{x}_i(\alpha_0)$.

- Let us take into account the above formula for the corresponding membership functions $m_i(x_i)$.

37.   Towards the resulting recommendation (cont-d)

- We can conclude that the narrowed intervals – over which we need to compute the range – have the following form:

$$\mathbf{x}_i(\alpha_0) = \{x_i : m_i(x_i) \leq \alpha_0\} = [\underline{x}_i + \alpha_0 \cdot (\widetilde{x}_i - \underline{x}_i), \overline{x}_i - \alpha_0 \cdot (\overline{x}_i - \widetilde{x}_i)].$$

- Thus, we arrive at the following recommendations on how to compute the narrowed alternative to the original too-wide interval $[\underline{y}, \overline{y}]$.

38.   Recommendation

- Suppose that we are given a function $f(x_1, \ldots, x_n)$ and intervals $[\underline{x}_1, \overline{x}_1]$, $\ldots$, $[\underline{x}_n, \overline{x}_n]$.

- To get a narrowed y-interval, we:

  - select some small value $\alpha_0 > 0$;

  - compute narrowed x-intervals

    $$\mathbf{x}_i(\alpha_0) = \{x_i : \mu_i(x_i) \leq \alpha_0\} = [\underline{x}_i + \alpha_0 \cdot (\widetilde{x}_i - \underline{x}_i), \overline{x}_i - \alpha_0 \cdot (\overline{x}_i - \widetilde{x}_i)], \text{ and}$$

  - use interval computations to find the range of $f(x_1, \ldots, x_n)$ on these narrowed intervals:

    $$[\underline{y}, \overline{y}] = f(\mathbf{x}_1(\alpha_0), \ldots, \mathbf{x}_n(\alpha_0)).$$

- We then conclude that the design is satisfactory is $\overline{y} \leq y_0$ and not satisfactory if $\overline{y} > y_0$.

39.   Mathematically justified approach: idea

- The main idea behind our approach is very simple.

- Computing the range of the function $f(x_1, \ldots, x_n)$ over the original intervals leads to a too wide range,

- So, we need to "narrow down" these intervals.

- So, we need to come up with an operation that would transform each interval into a narrower subinterval of the original interval.

- In other words, we need a function N that transform each interval $[\underline{x}, \overline{x}]$ into an interval $N([\underline{x}, \overline{x}]) \subset [\underline{x}, \overline{x}]$.

40. What are the natural properties of the desired narrowing function N?

- Usually, each number x_i is a value of some physical quantity – e.g., length or moment of time.

- We want to deal with values of a quantity.

- However, we actually deal with numbers representing these quantities.

- This is not just a philosophical difference.

- Numerical values change if we use a different measuring unit and/or a different starting point.

- If we replace a measuring unit by another unit which is a times smaller, then all numerical values get multiplied by a; $x \mapsto a \cdot x$.

- For example, if we replace meters with centimeters – which are 100 times smaller – then 1.7 m becomes $100 \cdot 1.7 = 170$ cm.

- We can also change sign – e.g., call positive electric charges negative and vice versa.

- Then all numerical values change sign, i.e., get multiplied by -1.

- If we replace the starting point with the one which is b units earlier, then the value b is added to all numerical values.

- E.g., 3 pm (measured starting with midday) becomes $3 + 12 = 15$ hours.

- These transformations do not change the physical situation.

- So, it is reasonable to require that the result of narrowing should not depend on what measuring unit and/or what starting point we use.

42. What are the natural properties of the desired narrowing function (cont-d)

- For example:
  - if we perform narrowing in the original units and then change the unit,
  - the results should be the same if we first change the unit and then perform the narrowing.
- Thus, we arrive at the following definition.

43. Definition and result

- *By a* narrowing operation, *we mean a function N that maps each interval $[\underline{x}, \overline{x}]$ into its proper subinterval $N([\underline{x}, \overline{x}]) \subset [\underline{x}, \overline{x}]$ for which:*

  - *for every $a > 0$, we have $N(a \cdot [\underline{x}, \overline{x}]) = a \cdot N([\underline{x}, \overline{x}])$;*
  - *for every b, we have $N([\underline{x}, \overline{x}] + b) = N([\underline{x}, \overline{x}]) + b$; and*
  - *we have $N(-[\underline{x}, \overline{x}]) = -N([\underline{x}, \overline{x}])$.*

- **Proposition.**

  - *Every narrowing operator has the following form, for some $\alpha_0 \in (0, 1]$:*

  $$N([\underline{x}, \overline{x}]) = [\underline{x}_i + \alpha_0 \cdot (\widetilde{x}_i - \underline{x}_i), \overline{x}_i - \alpha_0 \cdot (\overline{x}_i - \widetilde{x}_i)].$$

  - *Vice versa, each operation of this form is a narrowing operator.*

44.   Resulting recommendation

- First, we compute the narrowed intervals.

- Then, we compute the range $[\underline{y}, \overline{y}]$ of the original function $f(x_1, \ldots, x_n)$ over these intervals.

- We then claim that the design is acceptable if $\overline{y} \leq y_0$.

- *Comments.*

  - We can see that the mathematically justified method leads to exactly the same result as the heuristic fuzzy-based method.

  - So why do we need the heuristic explanation?

  - Because the existence of a commonsense explanation makes the results of the mathematically justified meth0d more convincing.

45. Proof

- Let us first prove that each narrowing operator has the desired form.

- Indeed, let N be a narrowing operator.

- Let us denote the endpoints of the interval $N([-1, 1])$ by α_- and α_+, so that $N([-1, 1]) = [\alpha_-, \alpha_+]$, with $-1 \leq \alpha_- \leq \alpha_+ \leq 1$.

- One can easily see that $-[-1, 1] = [-1, 1]$.

- So the third bullet in the definition of a narrowing operator implies that $-[\alpha_-, \alpha_+] = [\alpha_-, \alpha_+]$.

- By definition of $-[\underline{x}, \overline{x}]$, this means that $[-\alpha_+, -\alpha_-] = [\alpha_-, \alpha_+]$.

- By comparing the lower endpoints, we conclude that $\alpha_- = -\alpha_+$.

- Thus, the interval $N([-1, 1])$ has the form $N([-1, 1]) = [-\alpha_+, \alpha_+]$.

- Since $N([-1, 1])$ is a proper subinterval of the original interval $[-1, 1]$, we conclude that $\alpha_+ < 1$.

- Let us denote $\alpha_0 \stackrel{\text{def}}{=} 1 - \alpha_+$, so that $\alpha_+ = 1 - \alpha_0$.

46. Proof (cont-d)

- Then, $N([-1, 1]) = [-(1 - \alpha_0), 1 - \alpha_0]$.

- In terms of α_0, the inequality $\alpha_+ < 1$ takes the equivalent form $\alpha_0 > 0$.

- We have shown that the desired formula holds for the interval $[\underline{x}, \overline{x}] = [-1, 1]$.

- Let us now show that the desired formula holds for every interval $[\underline{x}, \overline{x}]$ with $\underline{x} < \overline{x}$.

- Let us apply the first bullet property with $a = \overline{x} - \widetilde{x} = \widetilde{x} - \underline{x}$ to the equality $N([-1, 1]) = [-(1 - \alpha_0), 1 - \alpha_0])$.

- Then, we conclude that

$$N([-(\widetilde{x} - \underline{x}), \overline{x} - \widetilde{x}]) = [-(\widetilde{x} - \underline{x}) \cdot (1 - \alpha_0), (\overline{x} - \widetilde{x}) \cdot (1 - \alpha_0)].$$

- The second bullet, with $b = \widetilde{x}$, now implies that

$$N([\underline{x}, \overline{x}] = [\widetilde{x} - (\widetilde{x} - \underline{x}) \cdot (1 - \alpha_0), \widetilde{x} + (\overline{x} - \widetilde{x}) \cdot (1 - \alpha_0)] =$$
$$[\underline{x}_i - \alpha_0 \cdot (\widetilde{x}_i - \underline{x}_i), \overline{x}_i + \alpha_0 \cdot (\overline{x}_i - \widetilde{x}_i)].$$

47. Proof (cont-d)

- The statement is proven.

- The inverse statement – that every operator of this form is a narrowing operator – is easy to directly check.

- The proposition is thus proven.

48. Acknowledgments