

Jacobian Conjecture: Making Sense of the AI-Assisted Mathematical Discovery

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1. What is a Jacobian and what is the Jacobian Conjecture

- This conjecture is about smooth (= differentiable) mappings $y = f(x)$ from $\mathbb{R}^n$ to itself:

$$y_1 = f_1(x_1, \dots, x_n), \dots, y_n = f_n(x_1, \dots, x_n).$$

- Specifically, it is about mappings in which all the functions $f_i(x_1, \dots, x_n)$ are polynomials.
- The simplest case is when all these polynomials are linear, i.e.,

$$f_i(x_1, \dots, x_n) = a_n + \sum_{j=1}^n b_{i,j} \cdot x_j.$$

- In this case, according to linear algebra, there are two options.
- The matrix b with components $b_{i,j}$ may be non-degenerate, i.e., its determinant $D = \det(b)$ is different from 0.
- In this case, the mapping is reversible.

2. What is a Jacobian and what is the Jacobian Conjecture (cont-d)

- The determinant may be equal to 0.
- In which case the mapping is not reversible, i.e., there exist tuples $x \neq x'$ for which $f(x) = f(x')$.
- What happens in a non-linear case?
- In this case, in the vicinity of each point $x^{(0)} = (x_1^{(0)}, \dots, x_n^{(0)})$, the mapping can be approximated by a linear transformation

$$y_i = f_i(x^{(0)}) + \sum_{j=1}^n b_{i,j}(x^{(0)}) \cdot (x_i - x_i^{(0)}), \text{ where}$$

$$b_{i,j}(x^{(0)}) \stackrel{\text{def}}{=} \frac{\partial f_i}{\partial x_j} \Big|_{x=x^{(0)}}.$$

- The matrix consisting of the elements b_{ij} is called the *Jacobian*.

3. What is a Jacobian and what is the Jacobian Conjecture (cont-d)

- In the linear case, for each point $x^{(0)}$, the Jacobian coincides with the matrix $b_{i,j}$: $b_{i,j}(x^{(0)}) = b_{i,j}$.
- So, in this case, the determinant $D(x^{(0)})$ of the Jacobian is a constant – it has the same value for all points $x^{(0)}$.
- Interesting, it turned out that:
 - for the case when $n = 2$,
 - every polynomial mapping with the constant non-zero determinant $D(x^{(0)})$ is reversible.
- For $n = 3$, for a long time, researchers could:
 - neither prove that every polynomial mapping with the constant non-zero determinant is reversible,
 - not could they find a counter-example.

4. What is a Jacobian and what is the Jacobian Conjecture (cont-d)

- The resulting hypothesis is that every polynomial mapping with constant non-zero determinant is reversible.
- This is known as the *Jacobian Conjecture*.

5. AI helped to find a counter-example

- In 2026, a researcher used AI to help find a counterexample – and such a counterexample was successfully found.
- In the equivalent form presented in another paper, this counterexample has the following form:

$$y_1 = x_1 - 3 \cdot x_1^2 \cdot x_2 - x_1^3 \cdot x_3;$$

$$y_2 = x_2 + 3 \cdot x_1 \cdot x_3 + 24 \cdot x_1 \cdot x_2^2 + 12 \cdot x_1^2 \cdot x_2 \cdot x_3 + 36 \cdot x_1^2 \cdot x_2^3 + 12 \cdot x_1^3 \cdot x_2^2 \cdot x_3;$$

$$y_3 = x_3 + 8 \cdot x_2^2 + 6 \cdot x_1 \cdot x_2 \cdot x_3 + 28 \cdot x_1 \cdot x_2^3 + 12 \cdot x_1^2 \cdot x_2^2 \cdot x_3 + 24 \cdot x_1^2 \cdot x_2^4 + 8 \cdot x_1^3 \cdot x_2^3 \cdot x_3.$$

6. Natural question

- As usual, AI does not provide any logical explanation of how it came up with this example.
- From this viewpoint, modern AI tools are black boxes: they produce results, but they do not explain how they got them.
- It is therefore desirable to make sense and to try to explain at least some of its features.

7. What is known

- It is known that the above transformation is invariant with respect to a special scaling $T_\lambda(x_1, x_2, x_3) = (\lambda \cdot x_1, \lambda^{-1} \cdot x_2, \lambda^{-2} \cdot x_3)$.
- Namely, for every tuple x and for every λ , if $y = f(x)$, then $T_\lambda(y) = f(T_\lambda(x))$.
- The scale-invariance condition helps explain why some monomials do not appear in the formula.
- But still there are infinitely many different monomials whose addition could preserve scale-invariance, e.g., $x_1^{3+n} \cdot x_2^n \cdot x_3$ in f_1 for $n = 1, 2, 3, \dots$
- We need additional explanations for the above formulas.

8. Let us divide this problem into two

- By explaining, we mean that we can have some simple conditions that would lead us to the above formula.
- It is natural to consider two steps:
- First, it would be nice to explain why these specific monomials appear in the formula and not any others.
- Once this is done, it would be nice to understand the numerical coefficients with which these monomials occur in the formula.
- In this paper, we provide a possible explanation for the first question – why these specific monomials.
- Specifically, we provide simple conditions that lead exactly to the monomials from the above formula.
- The second part is still an open problem.

9. Our Result

- *Let $f(x)$ be a scale-invariant polynomial mapping that has the following two properties:*
 - *for each i , one of the monomials with the highest overall degree in $f_i(x)$ has the form $x_1^2 \cdot x_2^{\alpha_2} \cdot x_3$ for some α_2 ;*
 - *for each i , the expression $f_i(x)$ is linear in x_3 .*
- *Then, for each i , all the monomials in $f_i(x)$ are exactly the monomials from the above formula.*
- Both conditions are necessary:
 - if we do not have the first condition, then, e.g., in $f_1(x)$, we could have additional monomials $x_1^{3+n} \cdot x_2^n \cdot x_3$ for $n = 1, 2, 3, \dots$;
 - if we do not have the second condition, then, e.g., in $f_3(x)$, we can have an additional monomial $x_2^2 \cdot x_3^2$.

10. Proof of the Proposition

- To prove this result, let us consider the cases $i = 1$, $i = 2$, and $i = 3$ one by one, starting with the case $i = 1$.
- A general monomial has the form $x_1^{\alpha_1} \cdot x_1^{\alpha_2} \cdot x_1^{\alpha_3}$ for some non-negative integers α_i .
- By definition, the overall degree d of the monomial is equal to the sum of the degrees α_i : $d = \alpha_1 + \alpha_2 + \alpha_3$.
- Scale-invariance implies that

$$(\lambda \cdot x_1)^{\alpha_1} \cdot (\lambda^{-1} \cdot x_2)^{\alpha_2} \cdot (\lambda^{-2} \cdot x_3)^{\alpha_3} = \lambda \cdot x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot x_3^{\alpha_3}.$$

- In the left-hand side, the dependence on λ has the form

$$\lambda^{\alpha_1} \cdot \lambda^{-\alpha_2} \cdot \lambda^{-2\alpha_3} = \lambda^{\alpha_1 - \alpha_2 - 2\alpha_3}.$$

- On the right-hand side, we have $\lambda = \lambda^1$. Thus, we must have

$$\alpha_1 - \alpha_2 - 2\alpha_3 = 1.$$

11. Case $i = 1$ (cont-d)

- In particular, this must be true for the term of the largest degree that has the form $x_1^3 \cdot x_2^{\alpha_2} \cdot x_3$.
- So for this term, we have $3 - \alpha_2 - 2 = 1$, so we must have $\alpha_2 = 0$.
- In this case, the overall degree of this term is $3 + 0 + 1 = 4$.
- Since this term has the largest overall degree, this means that all the monomials in $f_1(x)$ must have degree not larger than 4:

$$\alpha_1 + \alpha_2 + \alpha_3 \leq 4.$$

- Since the dependence of each term on x_3 is linear, we will always have $\alpha_3 = 0$ or $\alpha_3 = 1$. Let us consider these two cases one by one.

12. Case when $i = 1$ and $\alpha_3 = 0$

- Let us first consider the case when $\alpha_3 = 0$.
- In this case, the relation between α_i 's turns into $\alpha_1 - \alpha_2 = 1$, so $\alpha_1 = \alpha_2 + 1$.
- Substituting this expression for α_1 and 0 for α_3 into the inequality $\alpha_1 + \alpha_2 + \alpha_3 \leq 4$, we conclude that $2\alpha_2 + 1 \leq 4$.
- So $2\alpha_2 \leq 3$ and $\alpha_2 \leq 3/2$.
- Since α_2 is a non-negative integer, we have either $\alpha_2 = 0$ or $\alpha_2 = 1$.
- Let us consider these two cases one by one.
- If $\alpha_2 = 0$, then $\alpha_1 = \alpha_2 + 1 = 1$ and, since $\alpha_3 = 0$, we have the term x_1 .
- If $\alpha_2 = 1$, then $\alpha_1 = \alpha_2 + 1 = 2$, and, since $\alpha_3 = 0$, we have the term $x_1^2 \cdot x_2$.
- Both these terms are in $f_1(x)$.

13. Case when $i = 1$ and $\alpha_3 = 1$

- Let us now consider the case when $\alpha_3 = 1$.
- In this case, the formula $\alpha_1 - \alpha_2 - 2\alpha_3 = 1$ turns into $\alpha_1 - \alpha_2 - 2 = 1$, so $\alpha_1 = \alpha_2 + 3$.
- Substituting this expression for α_1 and 1 for α_3 into the inequality $\alpha_1 + \alpha_2 + \alpha_3 \leq 4$, we conclude that $2\alpha_2 + 3 \leq 4$, i.e., $2\alpha_2 \leq 1$ and $\alpha_2 \leq 1/2$.
- Since α_2 is a non-negative integer, we have $\alpha_2 = 0$.
- Then, $\alpha_1 = \alpha_2 + 3 = 3$ and, since $\alpha_3 = 0$, we have the term $x_1^3 \cdot x_2$.
- This term is also in $f_1(x)$.
- We have thus explained all the terms in $f_1(x)$.

14. Case when $i = 2$

- Similar to the $i = 1$ case, for terms in $f_2(x)$, scale-invariance implies that

$$\alpha_1 - \alpha_2 - 2\alpha_3 = -1.$$

- In particular, this must be true for the term of the largest degree that has the form $x_1^3 \cdot x_2^{\alpha_2} \cdot x_3$.
- For this term, we have $3 - \alpha_2 - 2 = -1$, so we must have $\alpha_2 = 2$.
- In this case, the overall degree of this term is $3 + 2 + 1 = 6$.
- Since this term has the largest overall degree, this means that all the monomials in $f_2(x)$ must have degree not larger than 6:

$$\alpha_1 + \alpha_2 + \alpha_3 \leq 6.$$

- Since the dependence of each term on x_3 is linear, we will always have $\alpha_3 = 0$ or $\alpha_3 = 1$. Let us consider these two cases one by one,

15. Case when $i = 2$ and $\alpha_3 = 0$

- In this case, $\alpha_1 - \alpha_2 - 2\alpha_3 = -1$ turns into $\alpha_1 - \alpha_2 = -1$, so $\alpha_2 = \alpha_1 + 1$.
- Substituting this expression for α_2 and 0 for α_3 into the inequality $\alpha_1 + \alpha_2 + \alpha_3 \leq 6$, we conclude that $2\alpha_1 + 1 \leq 6$.
- So, $2\alpha_1 \leq 5$ and $\alpha_1 \leq 5/2$. Since α_1 is a non-negative integer, we have either $\alpha_1 = 0$ or $\alpha_1 = 1$ or $\alpha_1 = 2$.
- Let us consider these three cases one by one.
- If $\alpha_1 = 0$, then $\alpha_2 = \alpha_1 + 1 = 1$ and, since $\alpha_3 = 0$, we have the term x_2 .
- If $\alpha_1 = 1$, then $\alpha_2 = \alpha_1 + 1 = 2$, and, since $\alpha_3 = 0$, we have the term $x_1 \cdot x_2^2$.
- If $\alpha_1 = 2$, then $\alpha_2 = \alpha_1 + 1 = 3$, and, since $\alpha_3 = 0$, we have the term $x_1^2 \cdot x_2^3$.
- All three terms are in $f_2(x)$.

16. Case when $i = 2$ and $\alpha_3 = 1$

- Let us now consider the case when $\alpha_3 = 1$.
- In this case, $\alpha_1 - \alpha_2 - 2\alpha_3 = -1$ turns into $\alpha_1 - \alpha_2 - 2 = -1$, so $\alpha_1 = \alpha_2 + 1$.
- Substituting this expression for α_1 and 1 for α_3 into the inequality $\alpha_1 + \alpha_2 + \alpha_3 \leq 6$, we conclude that $2\alpha_2 + 2 \leq 6$.
- So, $2\alpha_2 \leq 4$ and $\alpha_2 \leq 2$. Since α_2 is a non-negative integer, we can have $\alpha_2 = 0$, $\alpha_2 = 1$, or $\alpha_2 = 2$.
- Let us consider these three cases one by one.
- If $\alpha_2 = 0$, then $\alpha_1 = \alpha_2 + 1 = 1$ and, since $\alpha_3 = 1$, we have the term $x_1 \cdot x_3$.
- If $\alpha_2 = 1$, then $\alpha_1 = \alpha_2 + 1 = 2$, and, since $\alpha_3 = 1$, we have the term $x_1^2 \cdot x_2 \cdot x_3$.
- If $\alpha_2 = 2$, then $\alpha_1 = \alpha_2 + 1 = 3$, and, since $\alpha_3 = 1$, we have the term $x_1^3 \cdot x_2^2 \cdot x_3$.

17. Case when $i = 2$: conclusion

- All three terms are also in $f_2(x)$.
- We have thus explained all the terms in $f_2(x)$.

18. Case when $i = 3$

- Similar to the case $i = 1$, for terms in $f_3(x)$, scale-invariance implies that $\alpha_1 - \alpha_2 - 2\alpha_3 = -2$.
- In particular, this must be true for the term of the largest degree that has the form $x_1^3 \cdot x_2^{\alpha_2} \cdot x_3$.
- For this term, we have $3 - \alpha_2 - 2 = -2$, so we must have $\alpha_2 = 3$.
- In this case, the overall degree of this term is $3 + 3 + 1 = 7$.
- Since this term has the largest overall degree, this means that all the monomials in $f_3(x)$ must have degree not larger than 7:

$$\alpha_1 + \alpha_2 + \alpha_3 \leq 7.$$

- Since the dependence of each term on x_3 is linear, we will always have $\alpha_3 = 0$ or $\alpha_3 = 1$.
- Let us consider these two cases one by one.

19. Case when $i = 3$ and $\alpha_3 = 0$

- In this case, $\alpha_1 + \alpha_2 + \alpha_3 \leq 7$ turns into $\alpha_1 - \alpha_2 = -2$, so $\alpha_2 = \alpha_1 + 2$.
- Substituting this expression for α_2 and 0 for α_3 into the inequality $\alpha_1 + \alpha_2 + \alpha_3 \leq 7$, we conclude that $2\alpha_1 + 2 \leq 7$.
- So $2\alpha_1 \leq 5$ and $\alpha_1 \leq 5/2$. Since α_1 is a non-negative integer, we have either $\alpha_1 = 0$ or $\alpha_1 = 1$ or $\alpha_1 = 2$.
- Let us consider these three cases one by one.
- If $\alpha_1 = 0$, then $\alpha_2 = \alpha_1 + 2 = 2$ and, since $\alpha_3 = 0$, we have the term x_2^2 .
- If $\alpha_1 = 1$, then $\alpha_2 = \alpha_1 + 2 = 3$, and, since $\alpha_3 = 0$, we have the term $x_1 \cdot x_2^3$.
- If $\alpha_1 = 2$, then $\alpha_2 = \alpha_1 + 2 = 4$, and, since $\alpha_3 = 0$, we have the term $x_1^2 \cdot x_2^4$.
- All three terms are in $f_3(x)$.

20. Case when $i = 3$ and $\alpha_3 = 1$

- Let us now consider the case when $\alpha_3 = 1$.
- In this case, $\alpha_1 - \alpha_2 - 2\alpha_3 = -2$ turns into $\alpha_1 - \alpha_2 - 2 = -2$, so $\alpha_1 = \alpha_2$.
- Substituting this expression for α_1 and 1 for α_3 into the inequality $\alpha_1 + \alpha_2 + \alpha_3 \leq 7$, we conclude that $2\alpha_2 + 1 \leq 7$.
- So $2\alpha_2 \leq 6$ and $\alpha_2 \leq 3$. Since α_2 is a non-negative integer, we can have $\alpha_2 = 0$, $\alpha_2 = 1$, $\alpha_2 = 2$, or $\alpha_2 = 3$.
- Let us consider these three cases one by one.
- If $\alpha_2 = 0$, then $\alpha_1 = \alpha_2 = 0$ and, since $\alpha_3 = 1$, we have the term x_3 .
- If $\alpha_2 = 1$, then $\alpha_1 = \alpha_2 = 1$, and, since $\alpha_3 = 1$, we have the term $x_1 \cdot x_2 \cdot x_3$.
- If $\alpha_2 = 2$, then $\alpha_1 = \alpha_2 = 2$, and, since $\alpha_3 = 1$, we have the term $x_1^2 \cdot x_2^2 \cdot x_3$.

21. Case when $i = 3$ and $\alpha_3 = 1$ (cont-d)

- If $\alpha_2 = 3$, then $\alpha_1 = \alpha_2 = 3$, and, since $\alpha_3 = 1$, we have the term $x_1^3 \cdot x_2^3 \cdot x_3$.
- All three terms are also in $f_3(x)$.
- We have thus explained all the terms in $f_3(x)$.
- The proposition is proven.

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