How to Generate "Nice" Cubic Polynomials – with Rational Coefficients, Rational Zeros and Rational Extrema: A Fast Algorithm

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1. Need for Nice Calculus-Related Examples

- After students learn the basics of calculus, they practice them graphing functions $y = f(x)$.
- They find the roots (zeros), i.e., values where $f(x) = 0$.
- They find the extreme points, i.e., values where the derivative $f'(x)$ is equal to 0.
- They find out whether $f(x)$ increases or decreases between extreme points – by checking the sign of $f'(x)$.
- They use this information – plus the values of $f(x)$ at several points x – to graph the function.
- For this practice, students need examples for which they can compute both the zeros and the extreme points.

2. Cubic Polynomials: the Simplest Case When Such an Analysis Makes Sense

- The simplest possible functions are polynomials.
- For linear functions, the derivative is constant, so there are no extreme point.
- For quadratic functions, there is an extreme point.
- However, after studying quadratic equations, students already know how to graph the corresponding function.
- So, for quadratic polynomials, there is no need to use calculus.
- The simplest case when calculus tools are needed is the case of cubic polynomials.

3. To Make It Simpler For Students, It Is Desirable to Limit Ourselves to Rational Roots

- Students are much more comfortable with rational numbers than with irrational ones.
- Thus, it is desirable to have examples when all the coefficients, zeros, and extreme points of a are rational.
- Good news is that when we know that the roots are rational, it is (relatively) easy to find these roots.
- Indeed, for each rational root $x = p/q$ of a polynomial $a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_0$ with integer coefficients:
  - the numerator p is a factor of a_0, and
  - the denominator q is a factor of a_n.
- How can we find polynomials for which both zeros and extreme points are rational?

4. What Is Known and What We Do

- An algorithm for generating such polynomials was recently proposed.
- This algorithm, however, is not the most efficient one.
- For each tuple of the corresponding parameter values, it uses exhaustive trial-and-error search.
- In this presentation, we produce an efficient algorithm for producing nice polynomials.
- Namely, we propose simple computational formulas:
  - for each tuple of the corresponding parameters, these formulas produce a "nice" cubic polynomial;
  - every "nice" cubic polynomial can be thus generated.
- For each tuple, our algorithm requires the same constant number of elementary steps.

5. Analysis of the Problem

- A general cubic polynomial with rational coefficients has the form $a \cdot X^3 + b \cdot X^2 + c \cdot X + d$.
- Roots and extreme points of $f(x)$ do not change if we simply divide all its values by the same constant a.
- Thus, it is sufficient to consider polynomials with only three parameters: $X^3 + p \cdot X^2 + q \cdot X + r$, where
$$p \stackrel{\text{def}}{=} \frac{b}{a}, \quad q \stackrel{\text{def}}{=} \frac{c}{a}, \quad r \stackrel{\text{def}}{=} \frac{d}{a}.$$
- We can further simplify the problem if we replace X with $x \stackrel{\text{def}}{=} X + \dfrac{p}{3}$, then we get $x^3 + \alpha \cdot x + \beta$, where
$$\alpha = q - \frac{p^2}{3} \text{ and } \beta = r - \frac{p \cdot q}{3} + \frac{2p^3}{27}.$$
- Let r_1, r_2, and r_3 denote rational roots of $x^3 + \alpha \cdot x + \beta$, then, we have
$$x^3 + \alpha \cdot x + \beta = (x - r_1) \cdot (x - r_2) \cdot (x - r_3).$$
- So, $r_1 + r_2 + r_3 = 0$, $\alpha = r_1 \cdot r_2 + r_2 \cdot r_3 + r_1 \cdot r_3$, and $\beta = -r_1 \cdot r_2 \cdot r_3$.
- Substituting $r_3 = -(r_1 + r_2)$ into these formulas, we get
$$\alpha = -(r_1^2 + r_1 \cdot r_2 + r_2^2) \text{ and } \beta = r_1 \cdot r_2 \cdot (r_1 + r_2).$$

7. Using the Fact That the Extreme Points xo Should Also Be Rational

- Differentiating and equating the derivative to 0, we get
$$3x_0^2 - (r_1^2 + r_1 \cdot r_2 + r_2^2) = 0.$$
- This is equivalent to $3x_0^2 - 3y^2 - z^2 = 0$, where
$$y \stackrel{\text{def}}{=} \frac{r_1 + r_2}{2} \text{ and } z \stackrel{\text{def}}{=} \frac{r_1 - r_2}{2}.$$
- If we divide both sides of this equation by y^2, we get $3X_0^2 - 3 - Z^2 = 0$, where $X_0 \stackrel{\text{def}}{=} \dfrac{x_0}{y}$ and $Z \stackrel{\text{def}}{=} \dfrac{z}{y}$.
- One of the solution of above equation is easy to find: namely, when $X_0 = -1$, we get $Z^2 = 0$ and $Z = 0$.
- This means that for every y, $x_0 = -y$, y and $z = 0$ solve the above equation.
- We can now reconstruct r_1 and r_2 from y and z as $r_1 = y + z$ and $r_2 = y - z$,
- In our case, $r_1 = r_2 = y$, so $\alpha = -3y^2$ and $\beta = 2y^3$.
- We can then:
  - shift by a rational number s, ($x \to X = x + s$), and
  - multiply all the coefficients by an arbitrary rational number a.
- Then, we get
$$b = 3a \cdot s, \quad c = a \cdot (3s^2 - 3y^2), \quad d = a \cdot (s^3 + 2y^3).$$

9. Using the General Algorithm for Finding All Rational Solutions to a Quadratic Equation

- We have already found a solution of the equation $3X_0^2 - 3 - Z^2 = 0$, corresponding to $X_0 = -1$: then $Z = 0$.
- Every other solution (X_0, Z) can be connected to this simple solution $(-1, 0)$ by a straight line.
- A general equation of a straight line passing through the point $(-1, 0)$ is $Z = t \cdot (X_0 + 1)$.
- When X_0 and Z are rational, $t = \dfrac{Z}{X_0 + 1}$ is rational.
- Substituting this expression for Z into the equation, we get $3X_0^2 - 3 - t^2 \cdot (X_0 + 1)^2 = 0$.
- Since $X_0 \neq -1$, we can divide both sides by $X_0 + 1$. then $3 \cdot (X_0 - 1) - t^2 \cdot (X_0 + 1) = 0$, hence
$$X_0 = \frac{3 + t^2}{3 - t^2} \text{ and } Z = \frac{6t}{3 - t^2}.$$

10. Towards a General Description of All "Nice" Polynomial

- For every rational y, we can now take $x_0 = y \cdot X_0$, y, and $z = y \cdot Z = \dfrac{6yt}{3 - t^2}$.
- Based on y and z, we can compute $r_1 = y + z$ and $r_2 = y - z$.
- Then, we can compute α and β:
$$\alpha = -3y^2 - z^2 \text{ and } \beta = 2y \cdot (y^2 - z^2).$$
- Now, we can apply shift by s and multiplication by a.
- Thus, we arrive at the following algorithm for computing all possible "nice" cubic polynomials.

11. Resulting Algorithm for Computing All "Nice" Cubic Polynomials

- We use four arbitrary rational numbers t, y, s, and a; based on these numbers, we first compute $z = \dfrac{6yt}{3 - t^2}$.
- Then, we compute the coefficients b, c, and d of the resulting "nice" polynomial (a we already know):
$$b = 3a \cdot s; \quad c = a \cdot (3s^2 - 3y^2 - z^2);$$
$$d = a \cdot (s^3 + 2y \cdot (y^2 - z^2)).$$
- These expressions cover almost all "nice" polynomials, with the exception of the following family:
$$b = 3a \cdot s, \quad c = a \cdot (3s^2 - 3y^2), \quad d = a \cdot (s^3 + 2y^3).$$

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13. Bibliography

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