Use of Symmetries in Economics

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1. A Brief Overview

- Many semi-heuristic econometric formulas can be derived from the natural symmetry requirements.
- The list of such formulas includes many famous formulas provided by Nobel-prize winners, such as:
  - Hurwicz optimism-pessimism criterion for decision making under uncertainty,
  - McFadden's formula for probabilistic decision making,
  - Nash's formula for bargaining solution.
- It also includes Cobb-Douglas formula for production, gravity model for trade, etc.

2. How Do People Make Predictions?

- How do people make predictions? How did people know that the Sun will rise in the morning?
- Because in the past, the sun was always rising.
- In all these cases, to make a prediction, we look at similar situations in the past.
- We then make predictions based on what happened in such situations.
- Some predictions are more complicated than that – they are based on using physical laws.
- But how do we know that a law – e.g., Ohm's law – is valid?
- Because in several previous similar situations, this formula was true.

3. How to Describe This Idea in Precise Terms?

- We can shift or rotate the lab, Ohm's law will not change.
- In general, we have some phenomenon p depending on the situation s.
- We replace the original situation s by the changed situation $T(s)$.
- Invariance means that the phenomenon remains the same after the change: $p(T(s)) = p(s)$.
- A particular case of an invariance is when we have, e.g., a spherically symmetric object.
- If we rotate this object, it will remain the same – this is exactly what symmetry means in geometry.
- Because of this example, physicists call each invariance *symmetry*.

4. Symmetries Play a Fundamental Role in Physics

- In the past, physical theories – e.g., Newton's mechanics – were formulated in terms of diff. eq.
- Nowadays theories are usually formulated in terms of their symmetries, and equations can be derived.
- Traditional physical equations can also be derived from their symmetries.
- Predictions in economics are also based on similarity.
- So, let us see if we can derive economic equations from the corresponding symmetries.

5. Scaling

- Physical equations – like Ohm's law $V = I \cdot R$ – deal with numerical values of different physical quantities.
- These numerical values depend on the measuring unit.
- If we replace the original measuring unit with a λ times smaller one, then $x \to x' = \lambda \cdot x$.
- Fundamental equations $y = f(x)$ should not change if we change the measuring unit (e.g., dollars or pesos).
- We can't require $f(\lambda \cdot x) = f(x)$, then $f(x) = \text{const}$.
- We can require that for each λ, there is a $C(\lambda)$ s.t. if $y = f(x)$, then $y' = f(x')$, where $y' = C(\lambda) \cdot y$.
- For continuous $f(x)$, this implies $f(x) = A \cdot x^c$.
- The requirement $y = f(x_1, \ldots, x_n) \Rightarrow y' = f(x'_1, \ldots, x'_n)$, where $x'_i = \lambda_i \cdot x_i$ and $y' = C(\lambda_1, \ldots) \cdot y$, implies
$$f = A \cdot x_1^{c_1} \cdot \ldots \cdot x_n^{c_n}.$$

6. Shift

- For some quantities (e.g., time or temperature), the numerical value also depends on the starting point.
- If we replace the original starting point measuring unit with an earlier one, we get $x' = x + x_0$.
- Fundamental equations $y = f(x)$ should not change if we change the starting point.
- Example: salary itself? salary + social benefits?
- It's reasonable to require that for each x_0, there is a $C(x_0)$ s.t. $y = f(x) \Rightarrow y' = f(x')$, with $y' \stackrel{\text{def}}{=} C(\lambda) \cdot y$.
- For continuous $f(x)$, this implies $f(x) = A \cdot \exp(c \cdot x)$.

7. Additivity

- How trade y depends on the GDP x: $y = f(x)$?
- We can apply $f(x)$ to the whole country's GDP, or to regions whose GDPs are x' and x'': $x = x' + x''$.
- The result should be the same:
$$f(x' + x'') = f(x') + f(x'').$$
- For continuous $f(x)$, this implies $f(x) = c \cdot x$.
- In multi-D case, we have $f(x'_1 + x''_1, \ldots, x'_n + x''_n) = f(x'_1, \ldots, x'_n) + f(x''_1, \ldots, x''_n)$.
- This implies that $f(x_1, \ldots, x_n) = c_1 \cdot x_1 + \ldots + c_n \cdot x_n$.

8. How Can We Describe Human Preferences?

- We select a very good alternative A_+ and a very bad alternative A_-.
- For each $p \in [0, 1]$, $L(p)$ is a lottery in which we get A_+ with probability p, else A_-.
- For each realistic alternative A, it is better than $L(0) = A_-$ and worse than $L(1) = A_+$: $L(0) < A < L(1)$.
- Of course, if $L(p) < A$ and $p' < p$, then $L(p') < A$. Similarly, if $A < L(p)$ and $p < p'$, then $A < L(p')$.
- Thus, one can show that there exists a threshold value $u(A) = \sup\{p : L(p) < A\}$ (called *utility*) such that:
  - for $p < u(A)$, we have $L(p) < A$, and
  - for $p > u(A)$, we have $A < L(p)$.
- The alternative A is *equivalent* to the lottery $L(u(A))$, in the sense that $L(u - \varepsilon) < A < L(u + \varepsilon)$ for all $\varepsilon > 0$.

9. What If We Select Different A+ and A–?

- Let us consider the case when $A'_- < A_- < A_+ < A'_+$.
- Then, $A \sim L(u(A))$, i.e., A_+ with prob. $u(A)$ else A_-.
- $A_+ \sim L'(u'(A_+))$, i.e., A'_+ with prob. $u'(A_+)$ else A'_-.
- $A_- \sim L'(u'(A_-))$, i.e., A'_+ with prob. $u'(A_-)$ else A'_-.
- Thus, A is equivalent to a 2-step lottery in which we get A'_+ with probability
$$u'(A) = u(A) \cdot u'(A_+) + (1 - u(A)) \cdot u'(A_-).$$
- Otherwise, we get A'_-.
- Thus, changing a pair follows the same formulas as when we change the starting point and the meas. unit.
- Laws should not depend on the choice of a pair.
- So, we get scale- and shift-invariance.

10. How Utility u Depends on Money m

- For money, there is a natural starting point corresponding to 0 amount: no savings and no debts.
- Let us select a utility function for which this 0-money situation corresponds to 0 utility.
- Once the starting point is thus fixed, the only remaining utility transformation is scaling $u \to k \cdot u$.
- The numerical amount of money depends on the choice of the monetary unit.
- It is reasonable to require that the formula $u(m)$ does not change if we simply change the monetary unit.
- This scale-invariance leads to the power law $u = A \cdot m^c$.
- This is exactly what was experimentally observed, with $c \approx 0.5$.

11. Probabilistic Choice

- If we repeatedly offer the same choice to a person, he/she may make different choices in diff. iterations.
- The probability $p(a)$ of selecting an alternative grows with utility: $p(a) \sim f(u(a))$.
- The condition $\sum p(a) = 1$ implies $p(a) = f(u(a))/C$, where $C = \sum_a f(u(a))$.
- Utility is defined modulo shift.
- If we require that the probabilities do not change with shift, we get $f(u + u_0) = C(u_0) \cdot f(u)$.
- Thus, $f(u) = A \cdot \exp(c \cdot u)$.
- This is exactly the formula for which D. McFadden received his Nobel Prize in 2011.
- If we require scale-invariance, we get $f(u) = A \cdot u^c$ – which was also observed.

12. Decision Making under Interval Uncertainty

- In practice, we often only know the bounds $\underline{u}(a)$ and $\overline{u}(a)$ on the utility $u(a)$ of each alternative a.
- To make a decision under this uncertainty, we need to find an equivalent utility $u_0(\underline{u}, \overline{u})$.
- It's reasonable to require invariance: $u_0(\underline{u}, \overline{u}) = u \Rightarrow$
$$u_0(\underline{u} + \Delta u, \overline{u} + \Delta u) = u + \Delta u \text{ and } u_0(k \cdot \underline{u}, k \cdot \overline{u}) = k \cdot u.$$
- For $\alpha_H \stackrel{\text{def}}{=} u_0(0, 1)$ this implies $u_0(0, \overline{u} - \underline{u}) = (\overline{u} - \underline{u}) \cdot \alpha_H$ and $u_0(\underline{u}, \overline{u}) = \underline{u} + (\overline{u} - \underline{u}) \cdot \alpha_H$.
- This is exactly the formula for which Leo Hurwicz received his Nobel prize.
- Thus, Hurwicz's formula can be derived from natural symmetries.

13. Taking Future Effects into Account

- An option of getting \$1 at time t is less valuable than getting \$1 right away.
- What is the price $D(t)$ that a person should pay now for the option of getting \$1 at moment t?
- For any t and t_0, the value $D(t + t_0)$ can be estimated as follows:
  - \$1 at moment $t + t_0$ is worth $D(t)$ at moment t_0;
  - each dollar at moment t_0 is worth $D(t_0)$ now;
  - so we get $D(t_0) \cdot D(t)$.
- It is reasonable to require that this estimate also leads to $D(t + t_0)$: $D(t + t_0) = D(t_0) \cdot D(t)$.
- This is a particular case of shift-invariance, so $D(t) = \exp(c \cdot t)$.
- This is exactly the usual formula for discounting.

14. Group Decision Making: Nash's Idea

- The group may be unable to come to an agreement.
- The resulting situation is known as the *status quo*.
- We can shift each individual utility so that for the status quo solution, the utility of each participant is 0.
- The only remaining symmetries are $u_i \to u'_i = k_i \cdot u_i$.
- We want to combine n utilities $u_1, \ldots, u_n$ into a single utility value $u = f(u_1, \ldots, u_n)$.
- Scale-invariance $\Rightarrow f(u_1, \ldots, x_n) = A \cdot u_1^{c_1} \cdot \ldots \cdot u_n^{c_n}$.
- It is also reasonable to require that the decision should not change if we simply rename the participants.
- Thus, $c_i = \text{const}$, and $f(u_1, \ldots, u_n) = A \cdot (u_1 \cdot \ldots \cdot u_n)^c$.
- Maximizing $f \Leftrightarrow$ maximizing $u_1 \cdot \ldots \cdot u_n$ (Nash's idea).

15. Cobb-Douglas Production Function

- We know the country's overall capital K and overall labor input L.
- We want to estimate the country's production Y:
$$Y \approx f(K, L).$$
- Scale-invariance implies that $Y = A \cdot K^\alpha \cdot L^\beta$, for some α and β.
- This is exactly the well-known Cobb-Douglas production function.
- Thus, the Cobb-Douglas formula can also be derived from natural symmetries.

16. Gravity Model for Trade

- We want to estimate the trade volume t_{ij} between countries i, j based on GDPs g_i, g_j, and distance r_{ij}:
$$t_{ij} \approx f(g_i, g_j, r_{ij}).$$
- Additivity $\Rightarrow f$ is linear in g_i and g_j: $t_{ij} = g_i \cdot g_j \cdot H(r_{ij})$.
- The formulas should not change if we simply change the unit for distance.
- This scale-invariance implies that $H(r) = A \cdot r^c$.
- This is exactly the well-known gravity model.
- The usual gravity model only takes into account the GDPs g_i and g_j of the two countries.
- If we take into account populations p_i, p_j, we get:
$$t_{ij} = \frac{G_{gg} \cdot g_i \cdot g_j + G_{gp} \cdot g_i \cdot p_j + G_{pg} \cdot p_i \cdot g_j + G_{pp} \cdot p_i \cdot p_j}{r_{ij}^c}.$$

17. Linear ARMAX-GARCH Models

- We know the previous values X_{t-1}, X_{t-2}, …, of an economic quantity X.
- We also know the values d_t, d_{t-1}, …, of an external quantity d that affects X.
- We want to predict X_t: $X_t \approx f(X_{t-1}, X_{t-2}, \ldots, d_t, d_{t-1}, \ldots)$.
- For additive X and d (e.g., GDP X and investment d), additivity implies linearity:
$$X_t \approx \sum_{i=1}^{p} \varphi_i \cdot X_{t-i} + \sum_{i=1}^{b} \eta_i \cdot d_{t-i}.$$
- In a nutshell, this is exactly the AutoRegressive-Moving-Average model with eXogenous inputs (ARMAX).
- Thus, this model can indeed be justified by the corresponding symmetries.

18. Acknowledgments

This work was partially supported by the US National Science Foundation via grant HRD-1242122 (Cyber-ShARE).