# How Physics Can Influence What Is Computable: Taking Into Account that We Process Physical Data and that We Can Use Non-Standard Physical Phenomena to Process This Data

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# Part I
# Taking Into Account that We Process Physical Data

# 1. Computations with Real Numbers: Reminder

- From the physical viewpoint, real numbers $x$ describe values of different quantities.
- We get values of real numbers by measurements.
- Measurements are never 100% accurate, so after a measurement, we get an approximate value $r_k$ of $x$.
- In principle, we can measure $x$ with higher and higher accuracy.
- So, from the computational viewpoint, a real number is a sequence of rational numbers $r_k$ for which, e.g.,
$$|x - r_k| \le 2^{-k}.$$
- By an algorithm processing real numbers, we mean an algorithm using $r_k$ as an "oracle" (subroutine).
- This is how computations with real numbers are defined in *computable analysis*.

# 2. Known Negative Results

- No algorithm is possible that, given two numbers $x$ and $y$, would check whether $x = y$.
- Similarly, we can define a computable function $f(x)$ from real numbers to real numbers as a mapping that:
  - given an integer $n$, a rational number $x_m$ and its accuracy $2^{-m}$,
  - produces $y_n$ which is $2^{-n}$-close to all values $f(x)$ with $d(x, x_m) \le 2^{-m}$ (or nothing)

  so that for every $x$ and for each desired accuracy $n$, there is an $m$ for which a $y_n$ is produced.
- We can similarly define a computable function $f(x)$ on a computable compact set $K$.
- No algorithm is possible that, given $f$, returns $x$ s.t. $f(x) = \max\limits_{y \in K} f(y)$. (The max itself *is* computable.)

## 3. From the Physicists' Viewpoint, These Negative Results Seem Rather Theoretical

- In mathematics, if two numbers coincide up to 13 digits, they may still turn to be different.
- For example, they may be 1 and $1 + 10^{-100}$.
- In physics, if two quantities coincide up to a very high accuracy, it is a good indication that they are equal:
  - if an experimentally value is very close to the theoretical prediction,
  - this means that this theory is (triumphantly) true.
- This is how General Relativity was confirmed.
- This is how physicists realized that light is formed of electromagnetic waves: their speeds are very close.

# 4. How Physicists Argue

- In math, if two numbers coincide up to 13 digits, they may still turn to be different: e.g., 1 and $1 + 10^{-100}$.
- In physics, if two quantities coincide up to a very high accuracy, it is a good indication that they are equal.
- A typical physicist argument is that:
  - while numbers like $1 + 10^{-100}$ (or $c \cdot (1 + 10^{-100})$) are, in principle, possible,
  - they are *abnormal* (not *typical*).
- In physics, second order terms like $a \cdot \Delta x^2$ of the Taylor series can be ignored if $\Delta x$ is small, since:
  - while abnormally high values of $a$ (e.g., $a = 10^{40}$) are mathematically possible,
  - typical (= not abnormal) values appearing in physical equations are usually of reasonable size.

## 5. How to Formalize the Physicist's Intuition of Physically Meaningful Values: Main Idea

- To some physicist, all the values of a coefficient $a$ above 10 are abnormal.
- To another one, who is more cautious, all the values above 10 000 are abnormal.
- For every physicist, there is a value $n$ such that all value above $n$ are abnormal.
- This argument can be generalized as a following property of the set $\mathcal{T}$ of all physically meaningful elements.
- Suppose that we have a monotonically decreasing sequence of sets $A_1 \supseteq A_2 \supseteq \ldots$ for which $\bigcap\limits_n A_n = \emptyset$.
- In the above example, $A_n$ is the set of all numbers $\geq n$.
- Then, there exists an integer $N$ for which $\mathcal{T} \cap A_N = \emptyset$.

# 6. How to Formalize the Physicist's Intuition: Resulting Definition

- **Definition.** *We thus say that* $\mathcal{T}$ *is a* set of physically meaningful elements *if:*
  - *for every definable decreasing sequence* $\{A_n\}$ *for which* $\bigcap\limits_n A_n = \emptyset$,
  - *there exists an* $N$ *for which* $\mathcal{T} \cap A_N = \emptyset$.
- *Comment.* Of course, to make this definition precise,
  - we must restrict definability to a *subset* of properties,
  - so that the resulting notion of definability will be defined in ZFC itself.

## 7. Checking Equality of Real Numbers

- *Known:* equality of real numbers is undecidable.
- For physically meaningful real numbers, however, a deciding algorithm *is* possible:
  - *for every set* $\mathcal{T} \subseteq \mathbb{R}^2$ *which consists of physically meaningful pairs* $(x, y)$ *of real numbers,*
  - *there exists an algorithm deciding whether* $x = y$.
- *Proof:* We can take $A_n = \{(x, y) : 0 < |x - y| < 2^{-n}\}$. The intersection of all these sets is empty.
- Hence, $\mathcal{T}$ has no elements from $\bigcap\limits_{n=1}^{N_A} A_n = A_{N_A}$.
- Thus, for each $(x, y) \in \mathcal{T}$, $x = y$ or $|x - y| \geq 2^{-N_A}$.
- We can detect this by taking $2^{-(N_A+3)}$-approximations $x'$ and $y'$ to $x$ and $y$. Q.E.D.

## 8. Finding Roots

- In general, it is not possible, given a f-n $f(x)$ attaining negative and positive values, to compute its root.
- This becomes possible if we restrict ourselves to physically meaningful functions:
- *Let $K$ be a computable compact.*
- *Let $X$ be the set of all functions $f : K \to \mathbb{R}$ that attain 0 value somewhere on $K$. Then:*
  - *for every set $\mathcal{T} \subseteq X$ consisting of physically meaningful functions and for every $\varepsilon > 0$,*
  - *there is an algorithm that, given a f-n $f \in \mathcal{T}$, computes an $\varepsilon$-approximation to the set of roots*

$$R \stackrel{\text{def}}{=} \{x : f(x) = 0\}.$$

- In particular, we can compute an $\varepsilon$-approximation to one of the roots.

# 9. Optimization

- In general, it is not algorithmically possible to find $x$ where $f(x)$ attains maximum.
- Let $K$ be a computable compact. Let $X$ be the set of all functions $f : K \to \mathbb{R}$. Then:
  - *for every set* $\mathcal{T} \subseteq X$ *consisting of physically meaningful functions and for every* $\varepsilon > 0$,
  - *there is an algorithm that, given a f-n* $f \in \mathcal{T}$, *computes an* $\varepsilon$*-approx. to* $S = \left\{x : f(x) = \max\limits_y f(y)\right\}$.
- In particular, we can compute an approximation to an individual $x \in S$.
- *Reduction to roots:* $f(x) = \max\limits_y f(y)$ iff $g(x) = 0$, where $g(x) \stackrel{\text{def}}{=} f(x) - \max\limits_y f(y)$.

# 10. Computing Fixed Points

- In general, it is not possible to compute all the fixed points of a given computable function $f(x)$.
- Let $K$ be a computable compact. Let $X$ be the set of all functions $f : K \to K$. Then:
  - *for every set* $\mathcal{T} \subseteq X$ *consisting of physically meaningful functions and for every* $\varepsilon > 0$,
  - *there is an algorithm that, given a f-n* $f \in \mathcal{T}$, *computes an* $\varepsilon$*-approximation to the set* $\{x : f(x) = x\}$.
- In particular, we can compute an approximation to an individual fixed point.
- *Reduction to roots:* $f(x) = x$ iff $g(x) = 0$, where $g(x) \stackrel{\text{def}}{=} d(f(x), x)$.

# 11. Computing Limits

- *In general:* it is not algorithmically possible to find a limit $\lim a_n$ of a convergent computable sequence.
- Let $K$ be a computable compact. Let $X$ be the set of all convergent sequences $a = \{a_n\}$, $a_n \in K$. Then:
  - *for every set $\mathcal{T} \subseteq X$ consisting of physically meaningful functions and for every $\varepsilon > 0$,*
  - *there exists an algorithm that, given a sequence $a \in \mathcal{T}$, computes its limit with accuracy $\varepsilon$.*
- *Use:* this enables us to compute limits of iterations and sums of Taylor series (frequent in physics).
- *Main idea:* for every $\varepsilon > 0$ there exists $\delta > 0$ such that when $|a_n - a_{n-1}| \le \delta$, then $|a_n - \lim a_n| \le \varepsilon$.
- *Intuitively:* we stop when two consequent iterations are close to each other.

## Part II

# How to Take into Account that We Can Use Non-Standard Physical Phenomena to Process Data

## 12. Solving NP-Complete Problems Is Important

- In practice, we often need to find a solution that satisfies a given set of constraints.
- At a minimum, we need to check whether such a solution is possible.
- Once we have a candidate, we can feasibly check whether this candidate satisfies all the constraints.
- In theoretical computer science, "feasibly" is usually interpreted as computable in polynomial time.
- The class of all such problems is called NP.
- Example: satisfiability – checking whether a formula like $(v_1 \vee \neg v_2 \vee v_3) \,\&\, (v_4 \vee \neg v_2 \vee \neg v_5) \,\&\, \ldots$ can be true.
- Each problem from the class NP can be algorithmically solved by trying all possible candidates.

# 13. NP-Complete Problems (cont-d)

- For example, we can try all $2^n$ possible combinations of true-or-false values $v_1, \ldots, v_n$.
- For medium-size inputs, e.g., for $n \approx 300$, the resulting time $2^n$ is larger than the lifetime of the Universe.
- So, these exhaustive search algorithms are not practically feasible.
- It is not known whether problems from the class NP can be solved feasibly (i.e., in polynomial time).
- This is the famous open problem $\text{P} \stackrel{?}{=} \text{NP}$.
- We know that some problems are *NP-complete*: every problem from NP can be reduced to it.
- So, it is very important to be able to efficiently solve even one NP-hard problem.

## 14. Can Non-Standard Physics Speed Up the Solution of NP-Complete Problems?

- NP-complete means difficult to solve on computers based on the usual physical techniques.
- A natural question is: can the use of non-standard physics speed up the solution of these problems?
- This question has been analyzed for several specific physical theories, e.g.:
  - for quantum field theory,
  - for cosmological solutions with wormholes and/or casual anomalies.
- So, a scheme based on a theory may not work.

# 15. No Physical Theory Is Perfect

- If a speed-up is possible within a given theory, is this a satisfactory answer?
- In the history of physics,
  - always new observations appear
  - which are not fully consistent with the original theory.
- For example, Newton's physics was replaced by quantum and relativistic theories.
- Many physicists believe that every physical theory is approximate.
- For each theory $T$, inevitably new observations will surface which require a modification of $T$.
- Let us analyze how this idea affects computations.

# 16. No Physical Theory Is Perfect: How to Formalize This Idea

- *Statement:* for every theory, eventually there will be observations which violate this theory.
- To formalize this statement, we need to formalize what are *observations* and what is a *theory*.
- Most sensors already produce *observation* in the computer-readable form, as a sequence of 0s and 1s.
- Let $\omega_i$ be the bit result of an experiment whose description is $i$.
- Thus, all past and future observations form a (potentially) infinite sequence $\omega = \omega_1\omega_2\ldots$ of 0s and 1s.
- A physical *theory* may be very complex.
- All we care about is which sequences of observations $\omega$ are consistent with this theory and which are not.

# 17. What Is a Physical Theory?

- So, a physical theory $T$ can be defined as the set of all sequences $\omega$ which are consistent with this theory.
- A physical theory must have at least one possible sequence of observations: $T \neq \emptyset$.
- A theory must be described by a finite sequence of symbols: the set $T$ must be *definable.*
- How can we check that an infinite sequence $\omega = \omega_1\omega_2\ldots$ is consistent with the theory?
- The only way is check that for every $n$, the sequence $\omega_1\ldots\omega_n$ is consistent with $T$; so:

$$\forall n\, \exists \omega^{(n)} \in T\, (\omega_1^{(n)} \ldots \omega_n^{(n)} = \omega_1 \ldots \omega_n) \Rightarrow \omega \in T.$$

- In mathematical terms, this means that $T$ is *closed* in the Baire metric $d(\omega, \omega') \stackrel{\text{def}}{=} 2^{-N(\omega,\omega')}$, where

$$N(\omega, \omega') \stackrel{\text{def}}{=} \max\{k : \omega_1 \ldots \omega_k = \omega'_1 \ldots \omega'_k\}.$$

## 18. What Is a Physical Theory: Definition

- A theory must predict something new.
- So, for every sequence $\omega_1 \ldots \omega_n$ consistent with $T$, there is a continuation which does not belong to $T$.
- In mathematical terms, $T$ is *nowhere dense.*
- *By a* physical theory, *we mean a non-empty closed nowhere dense definable set* $T$.
- *A sequence* $\omega$ *is* consistent with the no-perfect-theory principle *if it does not belong to any physical theory.*
- In precise terms, $\omega$ does not belong to the union of all definable closed nowhere dense set.
- There are countably many definable set, so this union is *meager* (= *Baire first category*).
- Thus, due to Baire Theorem, such sequences $\omega$ exist.

# 19. How to Represent Instances of an NP-Complete Problem

- For each NP-complete problem $\mathcal{P}$, its instances are sequences of symbols.
- In the computer, each such sequence is represented as a sequence of 0s and 1s.
- We can append 1 in front and interpret this sequence as a binary code of a natural number $i$.
- In principle, not all natural numbers $i$ correspond to instances of a problem $\mathcal{P}$.
- We will denote the set of all natural numbers which correspond to such instances by $S_{\mathcal{P}}$.
- For each $i \in S_{\mathcal{P}}$, we denote the correct answer (true or false) to the $i$-th instance of the problem $\mathcal{P}$ by $s_{\mathcal{P},i}$.

# 20. What We Mean by Using Physical Observations in Computations

- In addition to performing computations, our computational device can:
  - produce a scheme $i$ for an experiment, and then
  - use the result $\omega_i$ of this experiment in future computations.
- In other words, given an integer $i$, we can produce $\omega_i$.
- In precise terms, the use of physical observations in computations means that use $\omega$ as an *oracle*.

## 21. Main Result

- A *ph-algorithm* $\mathcal{A}$ is an algorithm that uses an oracle $\omega$ consistent with the no-perfect-theory principle.
- The result of applying an algorithm $\mathcal{A}$ using $\omega$ to an input $i$ will be denoted by $\mathcal{A}(\omega, i)$.
- We say that a feasible ph-algorithm $\mathcal{A}$ *solves almost all instances of an NP-complete problem* $\mathcal{P}$ if:

$$\forall \varepsilon_{>0}\, \forall n\, \exists N_{\geq n} \left( \frac{\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\}}{\#\{i \leq N : i \in S_{\mathcal{P}}\}} > 1 - \varepsilon \right).$$

- Restriction to sufficiently long inputs $N \geq n$ makes sense: for short inputs, we can do exhaustive search.
- **Theorem.** *For every NP-complete problem $\mathcal{P}$, there is a feasible ph-alg. $\mathcal{A}$ solving almost all instances of $\mathcal{P}$.*

## 22. This Result Is the Best Possible

- Our result is the best possible, in the sense that the use of physical observations cannot solve *all* instances:
- **Proposition.** *If P≠NP, then no feasible ph-algorithm $\mathcal{A}$ can solve all instances of $\mathcal{P}$.*
- Can we prove the result for *all* $N$ starting with some $N_0$?
- We say that a feasible ph-algorithm $\mathcal{A}$ *$\delta$-solves* $\mathcal{P}$ if

$$\exists N_0\, \forall N \geq N_0 \left( \frac{\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\}}{\#\{i \leq N : i \in S_{\mathcal{P}}\}} > \delta \right).$$

- **Proposition.** *For every NP-complete problem $\mathcal{P}$ and for every $\delta > 0$:*
  - *if there exists a feasible ph-algorithm $\mathcal{A}$ that $\delta$-solves $\mathcal{P}$,*
  - *then there is a feasible algorithm $\mathcal{A}'$ that also $\delta$-solves $\mathcal{P}$.*

# Part III
# Physical and Computational Consequences

## 23. Justification of Physical Induction

- *What is physical induction:* a property $P$ is satisfied in the first $N$ experiments, then it is satisfied always.
- *Comment:* $N$ should be sufficiently large.
- *Theorem:* $\forall \mathcal{T}\, \exists N$ s.t. if for $o \in \mathcal{T}$, $P(o)$ is satisfied in the first $N$ experiments, then $P(o)$ is satisfied always.
- *Notation:* $s \stackrel{\text{def}}{=} s_1 s_2 \ldots$, where:
  - $s_i = T$ if $P(o)$ holds in the $i$-th experiment, and
  - $s_i = F$ if $\neg P(o)$ holds in the $i$-th experiment.
- *Proof:* $A_n \stackrel{\text{def}}{=} \{o : s_1 = \ldots = s_n = T \,\&\, \exists m\,(s_m = F)\}$; then $A_n \supseteq A_{n+1}$ and $\cup A_n = \emptyset$ so $\exists N\,(A_N \cap \mathcal{T} = \emptyset)$.
- *Meaning* of $A_N \cap \mathcal{T} = \emptyset$: if $o \in \mathcal{T}$ and $s_1 = \ldots = s_N = T$, then $\neg\exists m\,(s_m = F)$, i.e., $\forall m\,(s_m = T)$.

# 24. Ill-Posed Problem: Brief Reminder

- Main *objectives* of science:
  - *guaranteed* estimates for physical quantities;
  - *guaranteed* predictions for these quantities.
- *Problem:* estimation and prediction are ill-posed.
- *Example:*
  - measurement devices are inertial;
  - hence suppress high frequencies $\omega$;
  - so $\varphi(x)$ and $\varphi(x)+\sin(\omega\cdot t)$ are indistinguishable.
- *Existing approaches:*
  - statistical regularization (filtering);
  - Tikhonov regularization (e.g., $|\dot{x}|\le\Delta$);
  - expert-based regularization.
- *Main problem:* no guarantee.

## 25. On Physically Meaningful Solutions, Problems Become Well-Posed

- *State estimation – an ill-posed problem:*
  - *Measurement* $f$:
    state $s \in S \to$ observation $r = f(s) \in R$.
  - *In principle,* we can reconstruct $r \to s$:
    as $s = f^{-1}(r)$.
  - *Problem:* small changes in $r$ can lead to huge changes in $s$ ($f^{-1}$ *not continuous*).
- *Theorem:*
  - Let $S$ be a definably separable metric space.
  - Let $\mathcal{T}$ be a set of physically meaningful elements of $S$.
  - Let $f : S \to R$ be a continuous 1-1 function.
  - Then, the inverse mapping $f^{-1} : R \to S$
    is *continuous* for every $r \in f(\mathcal{T})$.

## 26. Everything Is Related: Einstein-Podolsky-Rosen (EPR) Paradox

- Due to *Relativity Theory*, two spatially separated simultaneous events cannot influence each other.
- *Einstein, Podolsky, and Rosen* intended to show that in quantum physics, such influence is possible.
- *In formal terms*, let $x$ and $x'$ be measured values at these two events.
- *Independence* means that possible values of $x$ do not depend on $x'$, i.e., $\mathcal{T} = X \times X'$ for some $X$ and $X'$.
- *Physical induction* implies that the pair $(x, x')$ belongs to a set $S$ of physically meaningful pairs.
- **Theorem.** *A set $\mathcal{T}$ os physically meaningful pairs cannot be represented as $X \times X'$.*

- Thus, everything *is* related – but we probably can't use this relation to pass information ($\mathcal{T}$ isn't computable).

## 27. When to Stop an Iterative Algorithm?

- *Situation* in numerical mathematics:
  - we often know an iterative process whose results $x_k$ are known to converge to the desired solution $x$,
  - but we do not know when to stop to guarantee that

$$d_X(x_k, x) \le \varepsilon.$$

- *Heuristic approach:* stop when $d_X(x_k, x_{k+1}) \le \delta$ for some $\delta > 0$.
- *Example:* in physics, if 2nd order terms are small, we use the linear expression as an approximation.

## 28. When to Stop an Iterative Algorithm: Result

- Let $\{x_k\} \in \mathcal{T}$, $k$ be an integer, and $\varepsilon > 0$ a real number.
- We say that $x_k$ is $\varepsilon$-*accurate* if $d_X(x_k, \lim x_p) \le \varepsilon$.
- Let $d \ge 1$ be an integer.
- By a *stopping criterion*, we mean a function $c : X^d \to R_0^+$ that satisfies the following two properties:
  - If $\{x_k\} \in \mathcal{T}$, then $c(x_k, \ldots, x_{k+d-1}) \to 0$.
  - If for some $\{x_n\} \in \mathcal{T}$ and $k$, $c(x_k, \ldots, x_{k+d-1}) = 0$, then $x_k = \ldots = x_{k+d-1} = \lim x_p$.
- *Result:* Let $c$ be a stopping criterion. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that
  - if $c(x_k, \ldots, x_{k+d-1}) \le \delta$, and the sequence $\{x_n\}$ is physically meaningful,
  - then $x_k$ is $\varepsilon$-accurate.

## Part IV

# Relation with Randomness

# 29. Towards Relation with Randomness

- If a sequence $s$ is random, it satisfies all the probability laws such as the law of large numbers.
- If a sequence satisfies all probability laws, then for all practical purposes we can consider it random.
- Thus, we can define a sequence to be random if it satisfies all probability laws.
- A probability law is a statement $S$ which is true with probability 1: $P(S) = 1$.
- So, a sequence is random if it belongs to all definable sets of measure 1.
- A sequence belongs to a set of measure 1 iff it does not belong to its complement $C = -S$ with $P(C) = 0$.
- So, *a sequence is* random *if it does not belong to any definable set of measure 0.*

## 30. Randomness and Kolmogorov Complexity

- Different definabilities lead to different randomness.
- When definable means computable, randomness can be described in terms of Kolmogorov complexity

$$K(x) \stackrel{\text{def}}{=} \min\{\text{len}(p) : p \text{ generates } x\}.$$

- Crudely speaking, an infinite string $s = s_1 s_2 \ldots$ is random if, for some constant $C > 0$, we have

$$\forall n \, (K(s_1 \ldots s_n) \geq n - C).$$

- Indeed, if a sequence $s_1 \ldots s_n$ is truly random, then the only way to generate it is to explicitly print it:

$$\texttt{print}(s_1 \ldots s_n).$$

- In contrast, a sequence like 0101...01 generated by a short program is clearly not random.

## 31. From Kolmogorov-Martin-Löf Theoretical Randomness to a More Physical One

- The above definition means that (definable) events with probability 0 cannot happen.
- In practice, physicists also assume that events with a *very small* probability cannot happen.
- For example, a kettle on a cold stove will not boil by itself – but the probability is non-zero.
- If a coin falls head 100 times in a row, any reasonable person will conclude that this coin is not fair.
- It is not possible to formalize this idea by simply setting a threshold $p_0 > 0$ below which events are not possible.
- Indeed, then, for $N$ for which $2^{-N} < p_0$, no sequence of $N$ heads or tails would be possible at all.

# 32. From Kolmogorov-Martin-Löf Theoretical Randomness to a More Physical One (cont-d)

- We cannot have a universal threshold $p_0$ such that events with probability $\le p_0$ cannot happen.
- However, we know that:
  - for each decreasing $(A_n \supseteq A_{n+1})$ sequence of properties $A_n$ with $\lim p(A_n) = 0$,
  - there exists an $N$ above which a truly random sequence cannot belong to $A_N$.
- *Resulting definition:* we say that $\mathcal{R}$ is a *set of random elements* if
  - for every definable decreasing sequence $\{A_n\}$ for which $\lim P(A_n) = 0$,
  - there exists an $N$ for which $\mathcal{R} \cap A_N = \emptyset$.

## 33. Random Sequences and Physically Meaningful Sequences

- Let $\mathcal{R}_K$ denote the set of all elements which are random in Kolmorogov-Martin-Löf sense. Then:
- *Every set of random elements consists of physically meaningful elements.*
- *For every set $\mathcal{T}$ of physically meaningful elements, the intersection $\mathcal{T} \cap \mathcal{R}_K$ is a set of random elements.*
- *Proof:* When $A_n$ is definable, for $D_n \stackrel{\text{def}}{=} \bigcap\limits_{i=1}^{n} A_i - \bigcap\limits_{i=1}^{\infty} A_i$, we have $D_n \supseteq D_{n+1}$ and $\bigcap\limits_{n=1}^{\infty} D_n = \emptyset$, so $P(D_n) \to 0$.
- Therefore, there exists an $N$ for which the set of random elements does not contain any elements from $D_N$.
- Thus, every set of random elements indeed consists of physically meaningful elements.

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# Part V
# Proofs

## 35. A Formal Definition of Definable Sets

- Let $\mathcal{L}$ be a theory.
- Let $P(x)$ be a formula from $\mathcal{L}$ for which the set $\{x \mid P(x)\}$ exists.
- We will then call the set $\{x \mid P(x)\}$ $\mathcal{L}$-*definable*.
- Crudely speaking, a set is $\mathcal{L}$-definable if we can explicitly *define* it in $\mathcal{L}$.
- All usual sets are definable: $\mathbb{N}$, $\mathbb{R}$, etc.
- Not every set is $\mathcal{L}$-definable:
  - every $\mathcal{L}$-definable set is uniquely determined by a text $P(x)$ in the language of set theory;
  - there are only countably many texts and therefore, there are only countably many $\mathcal{L}$-definable sets;
  - so, some sets of natural numbers are not definable.

## 36. How to Prove Results About Definable Sets

- Our objective is to be able to make mathematical statements about $\mathcal{L}$-definable sets. Therefore:
  - in addition to the theory $\mathcal{L}$,
  - we must have a stronger theory $\mathcal{M}$ in which the class of all $\mathcal{L}$-definable sets is a countable set.
- For every formula $F$ from the theory $\mathcal{L}$, we denote its Gödel number by $\lfloor F \rfloor$.
- We say that a theory $\mathcal{M}$ is *stronger* than $\mathcal{L}$ if:
  - $\mathcal{M}$ contains all formulas, all axioms, and all deduction rules from $\mathcal{L}$, and
  - $\mathcal{M}$ contains a predicate $\text{def}(n, x)$ such that for every formula $P(x)$ from $\mathcal{L}$ with one free variable,

$$\mathcal{M} \vdash \forall y\, (\text{def}(\lfloor P(x) \rfloor, y) \leftrightarrow P(y)).$$

## 37. Existence of a Stronger Theory

- As $\mathcal{M}$, we take $\mathcal{L}$ plus all above equivalence formulas.
- Is $\mathcal{M}$ consistent?
- Due to compactness, we prove that for any $P_1(x), \ldots, P_m(x)$, $\mathcal{L}$ is consistent with the equivalences corr. to $P_i(x)$.
- Indeed, we can take

$$\mathrm{def}(n, y) \leftrightarrow (n = \lfloor P_1(x) \rfloor \,\&\, P_1(y)) \vee \ldots \vee (n = \lfloor P_m(x) \rfloor \,\&\, P_m(y)).$$

- This formula is definable in $\mathcal{L}$ and satisfies all $m$ equivalence properties.
- Thus, the existence of a stronger theory is proven.
- The notion of an $\mathcal{L}$-definable set can be expressed in $\mathcal{M}$: $S$ is $\mathcal{L}$-definable iff $\exists n \in \mathbb{N}\, \forall y\, (\mathrm{def}(n, y) \leftrightarrow y \in S)$.
- So, all statements involving definability become statements from the $\mathcal{M}$ itself, *not* from metalanguage.

## 38. Consistency Proof

- *Statement:* $\forall \varepsilon > 0$, there exists a set $\mathcal{T}$ for which $\underline{P}(\mathcal{T}) \geq 1 - \varepsilon$.
- There are countably many definable sequences $\{A_n\}$: $\{A_n^{(1)}\}$, $\{A_n^{(2)}\}$, ...
- For each $k$, $P\left(A_n^{(k)}\right) \to 0$ as $n \to \infty$.
- Hence, there exists $N_k$ for which $P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k}$.
- We take $\mathcal{T} \stackrel{\text{def}}{=} -\bigcup\limits_{k=1}^{\infty} A_{N_k}^{(k)}$. Since $P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k}$, we have
$$\overline{P}\left(\bigcup_{k=1}^{\infty} A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} P\left(A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} \varepsilon \cdot 2^{-k} = \varepsilon.$$
- Hence, $\underline{P}(\mathcal{T}) = 1 - \overline{P}\left(\bigcup\limits_{k=1}^{\infty} A_{N_k}^{(k)}\right) \geq 1 - \varepsilon$.

## 39. Finding Roots: Proof

- To compute the set $R=\{x : f(x)=0\}$ with accuracy $\varepsilon>0$, let us take an $(\varepsilon/2)$-net $\{x_1,\ldots,x_n\}\subseteq K$.
- For each $i$, we can compute $\varepsilon'\in(\varepsilon/2,\varepsilon)$ for which $B_i\stackrel{\rm def}{=}\{x : d(x,x_i)\le\varepsilon'\}$ is a computable compact set.
- It is possible to algorithmically compute the minimum of a function on a computable compact set.
- Thus, we can compute $m_i\stackrel{\rm def}{=}\min\{|f(x)| : x\in B_i\}$.
- Since $f\in T$, similarly to the previous proof, we can prove that $\exists N\,\forall f\in T\,\forall i\,(m_i=0\vee m_i\ge 2^{-N})$.
- Comp. $m_i$ w/acc. $2^{-(N+2)}$, we check $m_i=0$ or $m_i>0$.
- Let's prove that $d_H(R,\{x_i : m_i=0\})\le\varepsilon$, i.e., that $\forall i\,(m_i=0\Rightarrow\exists x\,(f(x)=0\,\&\,d(x,x_i)\le\varepsilon))$ and $\forall x\,(f(x)=0\Rightarrow\exists i\,(m_i=0\,\&\,d(x,x_i)\le\varepsilon))$.

# 40. Finding Roots: Proof (cont-d)

- $m_i = 0$ means $\min\{|f(x)| : x \in B_i \stackrel{\text{def}}{=} B_{\varepsilon'}(x_i)\} = 0$.
- Since the set $K$ is compact, this value 0 is attained, i.e., there exists a value $x \in B_i$ for which $f(x) = 0$.
- From $x \in B_i$, we conclude that $d(x, x_i) \le \varepsilon'$ and, since $\varepsilon' < \varepsilon$, that $d(x, x_i) < \varepsilon$.
- Thus, $x_i$ is $\varepsilon$-close to the root $x$.
- Vice versa, let $x$ be a root, i.e., let $f(x) = 0$.
- Since the points $x_i$ form an $(\varepsilon/2)$-net, there exists an index $i$ for which $d(x, x_i) \le \varepsilon/2$.
- Since $\varepsilon/2 < \varepsilon'$, this means that $d(x, x_i) \le \varepsilon'$ and thus, $x \in B_i$.
- Therefore, $m_i = \min\{|f(x)| : x \in B_i\} = 0$. So, the root $x$ is $\varepsilon$-close to a point $x_i$ for which $m_i = 0$.

## 41. Proof of Well-Posedness

- *Known:* if a $f$ is continuous and 1-1 on a compact, then $f^{-1}$ is also continuous.
- *Reminder:* $S$ is compact if and only if it is closed and for every $\varepsilon$, it has a finite $\varepsilon$-net.
- *Given:* the set $X$ is definably separable.
- *Means:* $\exists$ def. $s_1, \ldots, s_n, \ldots$ everywhere dense in $X$.
- *Solution:* take $A_n \stackrel{\text{def}}{=} -\bigcup\limits_{i=1}^{n} B_\varepsilon(s_i)$.
- Since $s_i$ are everywhere dense, we have $\cap A_n = \emptyset$.
- Hence, there exists $N$ for which $A_N \cap \mathcal{T} = \emptyset$.
- Since $A_N = -\bigcup\limits_{i=1}^{N} B_\varepsilon(s_i)$, this means $\mathcal{T} \subseteq \bigcup\limits_{i=1}^{N} B_\varepsilon(s_i)$.
- Hence $\{s_1, \ldots, s_N\}$ is an $\varepsilon$-net for $\mathcal{T}$. Q.E.D.

## 42. Random Sequences and Physically Meaningful Sequences (proof cont-d)

- Let $T$ consist of physically meaningful elements. Let us prove that $\mathcal{T} \cap \mathcal{R}_K$ is a set of random elements.
- If $A_n \supseteq A_{n+1}$ and $P\left(\bigcap\limits_{n=1}^{\infty} A_n\right) = 0$, then for $B_m \stackrel{\text{def}}{=} A_m - \bigcap\limits_{n=1}^{\infty} A_n$, we have $B_m \supseteq B_{m+1}$ and $\bigcap\limits_{n=1}^{\infty} B_n = \emptyset$.
- Thus, by definition of a set consisting of physically meaningful elements, we conclude that $B_N \cap T = \emptyset$.
- Since $P\left(\bigcap\limits_{n=1}^{\infty} A_n\right) = 0$, we also know that $\left(\bigcap\limits_{n=1}^{\infty} A_n\right) \cap \mathcal{R}_K = \emptyset$.
- Thus, $A_N = B_N \cup \left(\bigcap\limits_{n=1}^{\infty} A_n\right)$ has no common elements with the intersection $T \cap \mathcal{R}_K$. Q.E.D.

# 43. Using Non-Standard Physics: Proof of the Main Result

- As $\mathcal{A}$, given an instance $i$, we simply produce the result $\omega_i$ of the $i$-th experiment.
- Let us prove, by contradiction, that for every $\varepsilon > 0$ and for every $n$, there exists an integer $N \geq n$ for which

$$\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, \omega_i = s_{\mathcal{P},i}\} > (1-\varepsilon) \cdot \#\{i \leq N : i \in S_{\mathcal{P}}\}.$$

- The assumption that this property is not satisfied means that for some $\varepsilon > 0$ and for some integer $n$, we have

$$\forall N_{\geq n}\, \#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, \omega_i = s_{\mathcal{P},i}\} \leq (1-\varepsilon) \cdot \#\{i \leq N : i \in S_{\mathcal{P}}\}.$$

- Let $T \stackrel{\text{def}}{=} \{x : \#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \leq$

$$(1-\varepsilon) \cdot \#\{i \leq N : i \in S_{\mathcal{P}}\} \text{ for all } N \geq n\}.$$

- We will prove that this set $T$ is a physical theory (in the sense of the above definition); then $\omega \notin T$.

# 44. Proof (cont-d)

- *Reminder:* $T = \{x : \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \le$
$$(1-\varepsilon)\cdot \#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N \ge n\}.$$
- By definition, a physical theory is a set which is non-empty, closed, nowhere dense, and definable.
- Non-emptiness is easy: the sequence $x_i = \neg s_{\mathcal{P},i}$ for $i \in S_{\mathcal{P}}$ belongs to $T$.
- One can prove that $T$ is closed, i.e., if $x^{(m)} \in T$ for which $x^{(m)} \to \omega$, then $x \in T$.
- Nowhere dense means that for every finite sequence $x_1 \ldots x_m$, there exists a continuation $x \notin T$.
- Indeed, for extension, we can take $x_i = s_{\mathcal{P},i}$ if $i \in S_{\mathcal{P}}$.
- Finally, we have an explicit definition of $T$, so $T$ is definable.

## 45. Non-Standard Physics: Proof of First Proposition

- Let us assume that P≠NP; we want to prove that for every feasible ph-algorithm $\mathcal{A}$, it is not possible to have

$$\forall N\,(\#\{i \le N : i \in S_{\mathcal{P}}\,\&\,\mathcal{A}(\omega, i) = s_{\mathcal{P},i}\} = \#\{i \le N : i \in S_{\mathcal{P}}\}).$$

- Let us consider, for each feasible ph-algorithm $\mathcal{A}$,

$$T(\mathcal{A}) \stackrel{\text{def}}{=} \{x : \#\{i \le N : i \in S_{\mathcal{P}}\,\&\,\mathcal{A}(x, i) = s_{\mathcal{P},i}\} = \#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N\}.$$

- Similarly to the proof of the main result, we can show that this set $T(\mathcal{A})$ is closed and definable.
- To prove that $T(\mathcal{A})$ is nowhere dense, we extend $x_1 \ldots x_m$ by 0s; then $x \in T$ would mean P=NP.
- If $T(\mathcal{A}) \ne \emptyset$, then $T(\mathcal{A})$ is a theory, so $\omega \notin T(\mathcal{A})$.
- If $T(\mathcal{A}) = \emptyset$, this also means that $\mathcal{A}$ does not solve all instances of the problem $\mathcal{P}$ – no matter what $\omega$ we use.

# 46. Proof of Second Proposition

- Let us assume that no non-oracle feasible algorithm $\delta$-solves the problem $\mathcal{P}$.
- Let's consider, for each $N_0$ and feasible ph-alg. $\mathcal{A}$,

$$T(\mathcal{A},N_0)\stackrel{\text{def}}{=}\{x:\#\{i\le N:i\in S_{\mathcal{P}}\,\&\,\mathcal{A}(x,i)=s_{\mathcal{P},i}\}>$$

$$\delta\cdot\#\{i\le N:i\in S_{\mathcal{P}}\}\text{ for all }N\ge N_0\}.$$

- We want to prove that $\forall N_0\,(\omega\notin T(\mathcal{A},N_0))$.
- Similarly to the proof of the Main Result, we can show that $T(\mathcal{A},N_0)$ is closed and definable.
- To prove that $T(\mathcal{A},N_0)$ is nowhere dense, we extend $x_1\ldots x_m$ by 0s.
- If $T(\mathcal{A},N_0)\ne\emptyset$, then $T(\mathcal{A},N_0)$ is a theory hence $\omega\notin T(\mathcal{A},N_0)$.
- If $T(\mathcal{A},N_0)=\emptyset$, then also $\omega\notin T(\mathcal{A},N_0)$.

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