# Type-2 Fuzzy Analysis Explains Ubiquity of Triangular and Trapezoid Membership Functions

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# 1. Membership Functions: Brief Reminder

- One of the main ideas behind fuzzy logic is:
  - to represent an imprecise ("fuzzy") natural-language property $P$ like "small"
  - by its *membership function* $\mu(x)$.

- Such a function assigns:
  - to each possible value $x$ of the corresponding property,
  - the degree $\mu(x) \in [0, 1]$ this value satisfies the property $P$ (e.g., to what extent $x$ is small).

# 2. Triangular and Trapezoid Membership Functions Are Ubiquitous: Why?

- According to this definition, we can have many different membership functions; however:

    - in many applications of fuzzy techniques,
    - the simplest piece-wise linear $\mu(x)$ – e.g., triangular and trapezoid ones – works very well.

- Why? In this talk, we use fuzzy techniques to analyze this question.

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# 3. How Can We Analyze the Problem: Need for a Type-2 Approach

- Traditionally – e.g., in control – fuzzy logic is used to select a value of a quantity, e.g., a control $u$.

- To come up with such a value, first, we use the experts' rules to come up,

  - for each possible control value $u$,
  - with a degree $d(u)$ to which this control value is reasonable.

- Then, we select a value $u$ – e.g., the one for which the degree of reasonableness is the largest:

$$d(u) \to \max_u.$$

- In our problem, instead of selecting a single *value* $u$, we select the whole membership *function* $\mu(x)$.

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# 4. Need for a Type-2 Approach (cont-d)

- To use fuzzy techniques for selecting $\mu$, we thus need to do the following.

- First, we need to use experts' rules to assign,
  - to each possible membership function $\mu(x)$,
  - a degree $d(\mu)$ to which this membership function is reasonable.

- Then, out of all possible members functions,
  - we select the one which is the most reasonable,
  - i.e., the one for which the degree of reasonableness $d(\mu)$ is the largest: $d(\mu) \to \max\limits_{\mu}$.

- Let us follow this path.

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# 5.   Comment

- Traditionally:

    - situations in which we use fuzzy to reason about real values is known as *type-1* fuzzy; while

    - situations in which we use fuzzy to reason about fuzzy is known as *type-2 fuzzy* approach.

- From this viewpoint, what we plan to use is an example of the type-2 fuzzy approach.

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# 6. Expert Rules

- First, we need to select expert rules.

- We consider the problem in its utmost generality.

- We want rules that will be applicable to all possible fuzzy properties.

- In this case, the only appropriate rule that comes to mind is the following natural natural-language rule:

  - if $x$ and $x'$ are close,

  - then $\mu(x)$ and $\mu(x')$ should be close.

- This rule exemplifies the whole idea of fuzziness:

  - for crisp properties (like $x \geq 0$), the degree of confidence abruptly changes the from 0 to 1;

  - instead, we have a smooth transition from 0 to 1.

# 7. How Can We Formalize This Expert Rule?

- There are infinitely many possible values of $x$ and $x'$.

- Thus, the above rule consists of infinitely many implications – one implication for each pair $(x, x')$.

- Dealing with *infinitely many* rules is difficult.

- It is therefore desirable to try to limit ourselves to *finite* number of rules.

- Such a limitation is indeed possible.

- Indeed, theoretically, we can consider all infinitely many possible values $x$.

- However, in practice, the values of any physical quantity are bounded: e.g.,
  - locations on the Earth are bounded by the Earth's diameter,
  - speeds are limited by the speed of light, etc.

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# 8. Formalizing the Expert Rule (cont-d)

- Thus, it is reasonable to assume that all possible values $x$ are within some interval $[\underline{x}, \overline{x}]$.

- Second, we only know $x$ and $x'$ with a certain accuracy $\varepsilon > 0$.

- From this viewpoint, there is no need to consider all infinitely many values.

- It is sufficient to consider only values on the grid of width $\varepsilon$, i.e., values

$$x_0 = \underline{x}, x_1 = \underline{x} + \varepsilon, x_2 = \underline{x} + 2\varepsilon, \ldots, x_n = \underline{x} + n \cdot \varepsilon = \overline{x}.$$

- So, it is sufficient to describe the values $\mu_i \stackrel{\text{def}}{=} \mu(x_i)$ of the membership function for $x_0, x_1, \ldots, x_n$.

- We call these values *discrete (d-)membership function*.

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## 9.  Formalizing the Expert Rule (cont-d)

- For these values, it is sufficient to formulate the "closeness" rule only for neighboring values $\mu_i$ and $\mu_{i+1}$:

  For all $i$, $\mu_i$ is close to $\mu_{i+1}$, i.e.

  ($\mu_1$ is close to $\mu_2$) and ... and (($\mu_{n-1}$ is close to $\mu_n$).

- This formula can be formalized according to the usual fuzzy methodology.

- Intuitively, closeness of $x$ and $x'$ is means that $d = |x - x'|$ is small.

- Thus, to express closeness, we need to select a membership function $s(d)$ describing "small".

- The larger the difference, the less small it is.

- So it is reasonable to require that the membership function $s(d)$ be strictly decreasing.

## 10.   Formalizing the Expert Rule (cont-d)

- At least, $s(d)$ decreases until it reaches value 0 for the differences $d$ which are clearly *not* small.

- $n$ is usually large, thus, $1/n$ is small.

- So, without losing generality, we can safely assume that the distance $1/n$ is small, i.e., that $s(1/n) > 0$.

- In terms of $s(d)$, for each $i$, the degree to which $\mu_i$ is close to $\mu_{i+1}$ is $s(|\mu_i - \mu_{i+1}|)$.

- To find the degree $d(\mu)$ to which a given d-membership function $\mu = (\mu_1, \ldots, \mu_n)$ is reasonable:

  - we need to apply some "and"-operation (t-norm) $f_\&(a, b)$ to these degrees,

  - then, $d(\mu) = f_\&(s(|\mu_0 - \mu_1|), \ldots, s(|\mu_{n-1} - \mu_n|))$.

# 11. Formalizing the Expert Rule (cont-d)

- It is reasonable to consider the simplest "and"-operation $f_{\&}(a, b) = \min(a, b)$, then we get

$$d(\mu) = \min(s(|\mu_0 - \mu_1|), \ldots, s(|\mu_{n-1} - \mu_n|)).$$

- Now, we are ready to formulate the problem in precise terms.

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## 12.    Definitions

- Let $n$ be a positive integer.

- Let $s(d)$ be a function from non-negative numbers to $[0,1]$ which is strictly increasing until it reaches 0 and for which $s(1/n) > 0$.

- By a *discrete (d-) membership function*, we mean a tuple $\mu = (\mu_0, \ldots, \mu_n)$.

- By a *degree of reasonableness $d(\mu)$* of a d-membership function $\mu$, we mean the value

$$d(\mu) = \min(s(|\mu_0 - \mu_1|), \ldots, s(|\mu_{n-1} - \mu_n|)).$$

- Let $M$ be a class of d-membership functions.

- We say that a d-membership function $\mu_{\text{opt}} \in M$ is *the most reasonable* d-membership function from $M$ if

$$d(\mu_{\text{opt}}) = \sup_{\mu \in M} d(\mu).$$

## 13.   Definitions

- Let $\mathcal{M}$ be a class of membership functions defined on an interval $[\underline{x}, \overline{x}]$.

- We say that a membership function $\mu(x) \in \mathcal{M}$ is *the most reasonable* membership function from $\mathcal{M}$ if:

  - for a sequence $n_k \to \infty$,
  - the corresponding d-membership functions are the most reasonable.

# 14. Main Results

- *Among all d-membership f-s for which $\mu_0 = 0$ and $\mu_n = 1$, the most reasonable is $\mu_i = \dfrac{i}{n}$.*

- Notice that our result does not depend on the selection of the membership function $s(d)$.

- *Among all $\mu(x)$ on $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = 0$ and $\mu(\overline{x}) = 1$, the most reasonable is $\mu(x) = \dfrac{x - \underline{x}}{\overline{x} - \underline{x}}$.*

- Thus, the most reasonable $\mu(x)$ is linear.

- *Among all d-membership f-s for which $\mu_0 = 1$ and $\mu_n = 0$, the most reasonable is $\mu_i = \dfrac{n - i}{n}$.*

- *Among all $\mu(x)$ on $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = 1$ and $\mu(\overline{x}) = 0$, the most reasonable is $\mu(x) = \dfrac{\overline{x} - x}{\overline{x} - \underline{x}}$.*

- Here also, the most reasonable $\mu(x)$ is linear.

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## 15.    Explaining Ubiquity of Trapezoid Functions

- Let us consider a property $P$ like "medium", for which:

  - the property $P$ is absolutely true for all values $x$ from some interval $[\underline{t}, \overline{t}]$, and

  - the property $P$ is absolutely false for all $x$ outside a wider interval $[\underline{T}, \overline{T}]$

- Such properties are common.

- In terms of membership degrees, the above condition means that:

  - $\mu(x) = 0$ for $x \leq \underline{T}$,

  - $\mu(x) = 1$ for $\underline{t} \leq x \leq \overline{t}$, and

  - $\mu(x) = 0$ for $x \geq \overline{T}$.

- On the intervals $[\underline{T}, \underline{t}]$ and $[\overline{t}, \overline{T}]$, we do not know the values of the membership function.

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# 16.  Ubiquity of Trapezoid Functions (cont-d)

- On both subintervals, it is reasonable to select the most reasonable membership function.

- We say that a d-membership function $\mu = (\mu_0, \ldots, \mu_n)$ is *normalized* if $\mu_i = 1$ for some $i$.

- *Among all normalized d-membership f-s for which $\mu_0 = \mu_{2k} = 0$, the most reasonable is the following one:*

  $-\ \mu_i = \dfrac{i}{k}$ *when $i \leq k$, and*

  $-\ \mu_i = \dfrac{2k - i}{k}$ *when $i \geq k$.*

- *Among all normalized $\mu(x)$ on $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = \mu(\overline{x}) = 0$, the most reasonable is:*

  $$\mu(x) = \frac{x - \widetilde{x}}{\widetilde{x} - \underline{x}} \text{ for } x \leq \widetilde{x} \text{ and } \mu(x) = \frac{\overline{x} - x}{\overline{x} - \widetilde{x}} \text{ for } x \geq \widetilde{x}.$$

- Thus, the most reasonable $\mu(x)$ is triangular.

# 17. How Robust Are These Results?

- How robust are these results?

- To answer this question, let us show that:

  - under two somewhat different approaches,

  - trapezoid and linear membership functions are still the most reasonable ones.

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## 18. What If We Use a Different "And"-Operation, e.g., Product?

- In the previous section, we used the min "and"-operation.

- What if we use a different "and"-operation – e.g., the algebraic product $f_\&(a, b) = a \cdot b$.

- This operation was also proposed by L. Zadeh in his original paper?

- In this case, the result depends, in general, on the selection of the membership function $s(d)$ for "small".

- All we know about "small" is that:
  - the value 0 is definitely absolutely small, and
  - there exists some value $D$ which is definitely not small.

## 19.    What If We Use Product (cont-d)

- This is one of the cases discussed in the previous section.

- So, let us use the results of the previous section to select the most reasonable membership function for small:

$$s_0(d) = 1 - \frac{d}{D} \text{ for } d \leq D \text{ and } s_0(d) = 0 \text{ for } d \geq D.$$

- For this selection, we get the following results.

- Let $n$ be a positive integer, and let $D > 0$ be a positive real number.

- Let $s_0(d) = 1 - \frac{d}{D}$ for $d \leq D$ and $s_0(d) = 0$ for $d \geq D$.

- By a *product-based degree of reasonableness* $d_0(\mu)$ of a d-membership function $\mu$, we mean the value

$$d_0(\mu) = s_0(|\mu_0 - \mu_1|) \cdot \ldots \cdot s_0(|\mu_{n-1} - \mu_n|).$$

## 20.   What If We Use Product (cont-d)

- Let $M$ be a class of d-membership functions.

- We say that $\mu_{\mathrm{opt}} \in M$ is *the most product-based reasonable* if $d_0(\mu_{\mathrm{opt}}) = \sup\limits_{\mu \in M} d_0(\mu)$.

- Let $\mathcal{M}$ be a class of membership functions defined on an interval $[\underline{x}, \overline{x}]$.

- We say that $\mu(x) \in \mathcal{M}$ is *the most product-based reasonable* on $\mathcal{M}$ if:

  - for a sequence $n_k \to \infty$,
  - the corresponding d-membership functions are the most product-based reasonable.

- *Among all d-membership f-s for which $\mu_0 = 0$ and $\mu_n = 1$, the most product-based reasonable is $\mu_i = \dfrac{i}{n}$.*

## 21.    What If We Use Product (cont-d)

- *Among all $\mu(x)$ on $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = 0$ and $\mu(\overline{x}) = 1$, the most product-based reasonable is*

$$\mu(x) = \frac{x - \underline{x}}{\overline{x} - \underline{x}}.$$

- Thus, the most reasonable membership function is linear.

- *Among all d-membership f-s for which $\mu_0 = 1$ and $\mu_n = 0$, the most product-based reasonable is $\mu_i = \dfrac{n - i}{n}$.*

- *Among all $\mu(x)$ on $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = 1$ and $\mu(\overline{x}) = 0$, the most product-based reasonable is*

$$\mu(x) = \frac{\overline{x} - x}{\overline{x} - \underline{x}}.$$

- Thus, here also, the most reasonable membership function is linear.

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# 22.    What If We Use Product (cont-d)

- Similarly to the previous section, this explains the ubiquity of trapezoid membership functions.

- *Among all normalized $\mu_i$ for which $\mu_0 = \mu_{2k} = 0$, the most product-based reasonable is:*

$$\mu_i = \frac{i}{k} \text{ when } i \leq k \text{ and } \mu_i = \frac{2k - i}{k} \text{ when } i \geq k.$$

- *Among all normalized $\mu(x)$ on $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = \mu(\overline{x}) = 0$, the most product-based reasonable is:*

$$\mu(x) = \frac{x - \widetilde{x}}{\widetilde{x} - \underline{x}} \text{ for } x \leq \widetilde{x} \text{ and } \mu(x) = \frac{\overline{x} - x}{\overline{x} - \widetilde{x}} \text{ for } x \geq \widetilde{x}.$$

- Thus, here also, the most reasonable membership function is a triangular one.

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## 23. What If We Use Statistics-Motivated Least Squares Approach to Select $\mu(x)$

- We used fuzzy techniques to determine:
  - the degree to which a d-membership function is reasonable,
  - i.e., a degree to which $\mu_1 - \mu_0$ is small, $\mu_2 - \mu_1$ is small, etc.

- Intuitively, small means close to 0, i.e., being approximately equal to 0.

- In other words, we determine a degree to which the following system of approximate equalities hold:

$$\mu_1 - \mu_0 \approx 0, \ldots, \mu_n - \mu_{n-1} \approx 0.$$

- It is worth noticing that such systems of approximate equation are well known in statistics.

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## 24.　Statistics Approach (cont-d)

- In statistics, the usual way of dealing with such system is to use the Least Squares approach.

- So, we look for the solutions for which the sum of the squares of the approximation errors is the smallest:

$$(\mu_1 - \mu_0)^2 + \ldots + (\mu_n - \mu_{n-1})^2 \to \min.$$

- Thus, we arrive at the following definitions.

- Let $n$ be a positive integer.

- By the *least-squares degree of reasonableness* $d_1(\mu)$ of a d-membership function $\mu$, we mean the value

$$d_1(\mu) = (\mu_0 - \mu_1)^2 + \ldots + (\mu_{n-1} - \mu_n)^2.$$

- Let $M$ be a class of d-membership functions.

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## 25.   Statistics Approach (cont-d)

- We say that $\mu_{\text{opt}} \in M$ is *the most least-squares reasonable* in $M$ if $d_1(\mu_{\text{opt}}) = \sup\limits_{\mu \in M} d_1(\mu)$.

- Let $\mathcal{M}$ be a class of membership functions defined on an interval $[\underline{x}, \overline{x}]$.

- We say that a membership function $\mu(x) \in \mathcal{M}$ is *the most least-squares reasonable reasonable* in $\mathcal{M}$ if:

  – for a sequence $n_k \to \infty$,

  – the corresponding d-membership functions are the most least-squares reasonable.

- *Among all d-membership f-s for which $\mu_0 = 0$ and $\mu_n = 1$, the most least-squares reasonable is $\mu_i = \dfrac{i}{n}$.*

## 26.    Statistics Approach (cont-d)

- *Among all $\mu(x)$ on $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = 0$ and $\mu(\overline{x}) = 1$, the most least-squares reasonable is*

$$\mu(x) = \frac{x - \underline{x}}{\overline{x} - \underline{x}}.$$

- Thus, the most reasonable $\mu(x)$ is linear.

- *Among all d-membership f-s for which $\mu_0 = 1$ and $\mu_n = 0$, the most least-squares reasonable is $\mu_i = \dfrac{n - i}{n}$.*

- *Among all $\mu(x)$ on $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = 1$ and $\mu(\overline{x}) = 0$, the most least-squares reasonable is*

$$\mu(x) = \frac{\overline{x} - x}{\overline{x} - \underline{x}}.$$

- Thus, here also, the most reasonable membership function is linear.

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## 27.    Statistics Approach (cont-d)

- Similarly to the previous section, this explains the ubiquity of trapezoid membership functions.

- *Among all normalized $\mu_i$ for which $\mu_0 = \mu_{2k} = 0$, the most least-squares reasonable is:*

$$\mu_i = \frac{i}{k} \text{ when } i \leq k \text{ and } \mu_i = \frac{2k-i}{k} \text{ when } i \geq k.$$

- *Among all normalized $\mu(x)$ on $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = \mu(\overline{x}) = 0$, the most least-squares reasonable is:*

$$\mu(x) = \frac{x - \widetilde{x}}{\widetilde{x} - \underline{x}} \text{ for } x \leq \widetilde{x} \text{ and } \mu(x) = \frac{\overline{x} - x}{\overline{x} - \widetilde{x}} \text{ for } x \geq \widetilde{x}.$$

- Thus, here also, the most reasonable membership function is a triangular one.

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# 28.  Acknowledgments

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# 29.   Proofs: Main Ideas

- By selecting $\mu_0 < \mu_1 < \ldots < \mu_n = 1$, we represent 1 as the sum of $n$ differences $\mu_i - \mu_{i-1}$.

- If some differences are different, one of them is $d > 1/n$, thus, $s(d) < s(1/n)$.

- So, the smallest of the values $s(\mu_i - \mu_{i-1})$ is $< s(1/n)$.

- Thus, the largest possible value $s(1/n)$ of $d(\mu)$ is attained when all the differences are the same, i.e., when

$$\mu_i = \frac{i}{n}.$$

- Similar result holds for the trapezoid case.

# 30. Proofs (cont-d)

- For the product, we use the result that the geometric mean is $\leq$ arithmetic mean.

- This means that the product is $\leq$ the $n$-th power of the arithmetic mean.

- Thus, the product of the values $s(\mu_i - \mu_{i-1})$ is $\leq$ what we get when all the differences are equal.

- So, the optimal solution is when all differences are equal: $\mu_i = i/n$.

- For Least Squares, differentiating w.r.t. $\mu_i$ and equating derivative to 0, we get

$$(\mu_i - \mu_{i+1}) + (\mu_i - \mu_{i-1}) = 0.$$

- This implies that $\mu_{i+1} - \mu_i = \mu_i - \mu_{i-1}$, so all differences are equal and $\mu_i = i/n$.