Different Concepts, Similar Computational Complexity: Nguyen's Results about Fuzzy and Interval Computations 35 Years Later

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1. Emergence of modern science

- In the ancient times, there was no clear separation between speculations, prejudices, feeling, and confirmed scientific facts; for example:
  - Johannes Kepler made great discoveries in astronomy – in particular, he discovered that planets follow elliptical orbits,
  - but he also described horoscopes predicting people's fate.
- Actually, his salaried position required him to deal both with astronomy and astrology – which at that time, were not clearly separated.
- Chemists were:
  - analyzing chemical reactions,
  - and at the time, tried different magical incantations that would help them turn matter into gold, and
  - they were paid for doing both chemistry and alchemy.

2. Emergence of modern science (cont-d)

- The situation gradually changed when the need for reliable practical applications necessitates a clear separation between:
  - science – that studies well-established facts and relations – and
  - semi-poetic imprecise speculations and feeling.
- The great Newton was interpreting the Bible, the great Goethe came up with his theory of vision.
- However, these activities were clearly outside what was then considered as science.

3. Traditional scientific approach to uncertainty

- Of course, usually, no matter what object we study, we do not have a complete knowledge of this object.
- In science, knowledge about objects is usually described in terms of numbers.
- For example, in mechanics, each object is characterized:
  - by its mass,
  - by the coordinates of its spatial location,
  - by the components of its velocity, and
  - if this object is rotating – by a unit vector describing its rotation axis and by the angular velocity.
- The fact that we do not have a complete knowledge means that we do not know the exact values of each of these quantities x.

4. Traditional scientific approach to uncertainty (cont-d)

- At best, we know the bounds $\underline{x}$ and $\overline{x}$ on the actual value.
- In such situations, all we know about the actual (unknown) value x is that this value belongs to the interval $[\underline{x}, \overline{x}]$.
- In some cases, we know that some values inside this interval are not possible.
- In this case, the set of all possible values of x has a more general form than an interval.
- But even in this case:
  - for some real numbers, we are, at this stage, 100% sure that the value of the physical quantity x cannot be equal to this number,
  - while for other numbers, we are 100% sure that the value of the quantity x can be equal to this number.

5. Traditional scientific approach to uncertainty (cont-d)

- We may have an additional gut feeling that some numbers from this interval are more possible than others.
- However, such gut feeling was not taken into account in the traditional scientific paradigm.
- Sometimes:
  - in addition to the interval (or, more generally, set) of possible values of x,
  - we also have some information about the frequency of different numbers from this interval.
- But again, what traditional science considered was only guaranteed knowledge about these probabilities.

6. Need to go beyond traditional scientific paradigm

- The paradigm change started in the early 1960s, with Lotfi Zadeh:
  - one of the world's leading specialists in automatic control,
  - a co-author of the then most widely used book on automatic control.
- He noticed that in many practical situations:
  - automatic controllers – that take into account all scientific information about the object of control
  - perform much worse than human controllers.
- He realized that human expert controllers use additional knowledge.
- This additional knowledge is – in contrast to what traditional science considered – not precise.

7. Need to go beyond traditional scientific paradigm (cont-d)

- The rules describing this knowledge were usually formulated:
  - in terms of imprecise ("fuzzy") words from natural language, such as "small", "approximately", etc.,
  - words that the traditional scientific approach ignored.
- So, instead of ignoring these words, Zadeh proposed to incorporate the corresponding scientific knowledge into the automatic controllers.
- For this purpose, he came up with a methodology that he called *fuzzy*.

8. Fuzzy methodology: a brief description

- For precise ("crisp") properties like "smaller than 10", every number either satisfies this property or does not.
- Such properties can be described by describing a set of all the values that satisfy this property.
- Equivalently, they can be described by a function $\mu(x)$ that assigns:
  - to each possible value x,
  - the value 1 if x has this property and 0 if not.
- In mathematics, such a function is known as a *characteristic function*.
- In contrast, for fuzzy words like "small":
  - for some values x,
  - experts are not 100% sure whether x is small or not,
  - they are only sure to some degree.
- We want to process such information in a computer.

9. Fuzzy methodology: a brief description (cont-d)

- Computers were not designed to process words from natural languages.
- They were designed to process numbers.
- So, to be able to use computers to process expert information, we need to be able to describe this degree of confidence by a number.
- A natural way to do it is to ask the expert to mark his/her degree on a scale from 0 to 1, where:
  - 1 means absolute confidence, and
  - 0 means no confidence at all.
- Alternatively, we can use a scale from 0 to 10 or from 0 to 5, and then divide the result, correspondingly, by 10 or by 5.
- This is how students evaluate their instructors, this is how we evaluate the quality of different services.

10. Fuzzy methodology: a brief description (cont-d)

- Thus, to describe a property, we need to describe a function that assigns:
  - to each value x,
  - a degree $\mu(x) \in [0, 1]$ to which each value x satisfies the given property (e.g., is small).
- The corresponding function is known as a *membership function* or, alternatively, as a *fuzzy set*.
- An additional complication comes from the fact that many rules describe what happens when several properties are satisfied.
- For example, a rule may describe what to do if:
  - the temperature t in a chemical reactor is slightly below the desired one *and*
  - the pressure p is slightly higher than desired.

11. Fuzzy methodology: a brief description (cont-d)

- We can, in principle, ask the expert to mark:
  - the degrees $\mu_{\text{temp}}(t)$ corresponding to different values t and
  - the degrees $\mu_{\text{press}}(p)$ corresponding to different values p.
- However, what we real need is, for all possible pairs (t, p), to estimate the degree of the above "and"-statement.
- It may be still possible to ask the expert about all such pairs.
- However, what if there are 5 inputs? Ten inputs?
- Even if we consider only 10 different values for each quantity, this would still make 10^5 or even 10^{10} combinations.
- We cannot ask that many questions to an expert.
- So, we cannot always directly elicit the expert's degree of certainty in an "and"-statement – of the type $A \,\&\, B$.

12. Fuzzy methodology: a brief description (cont-d)

- Thus, we need to be able to estimate this degree:
  - based on whatever information we have,
  - i.e., based on the degrees of certainty a and b of the statements A and B.
- The algorithm for this estimation is known as an *"and"-operation* (or, for historical reasons, a *t-norm*).
- We will denote it by $f_\&(a, b)$.
- The most widely used "and"-operations are $\min(a, b)$ and $a \cdot b$.
- Similarly, we need an algorithm to estimate the degree of certainty of $A \vee B$.
- Such an algorithm is known as an *"or"-operation*, or a *t-conorm*.
- We will denote it by $f_\vee(a, b)$.
- The most widely used "or"-operations are $\max(a, b)$ and $a + b - a \cdot b$.

13. Fuzzy methodology: a brief description (cont-d)

- To describe the degree of certainty of a negation $\neg A$, we need an algorithm which is known as a *negation operation* $f_\neg(a)$.
- The mostly widely used negation operation is $f_\neg(a) = 1 - a$.
- Because of the important role of these logical operations, fuzzy methodology is often called *fuzzy logic methodology*.

14. Need for data processing

- In many real-life situations, we are interesting in quantities which cannot be measured directly – e.g., in future values of some quantities.
- We cannot measure these quantities y directly.
- So, we need to estimate y:
  - based on available information,
  - i.e., based on the known values $\widetilde{x}_1, \ldots, \widetilde{x}_n$ of related quantities

$$x_1, \ldots, x_n.$$

- We will denote the estimating algorithm by $y = f(x_1, \ldots, x_n)$.

15. Need to take uncertainty into account

- The values $\widetilde{x}_i$ come either from measurements or from expert estimates.
- In both cases, the available value $\widetilde{x}_i$ is somewhat different from the actual (unknown) value x_i of the corresponding quantity.
- As a consequence:
  - even if the relation $y = f(x_1, \ldots, x_n)$ is exact,
  - the resulting estimate $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ for y is, in general, different from the actual value y of the quantity of interest.
- To make decisions, we need to know how accurate is this estimate.

16. Case of interval uncertainty

- In the interval case:
  - for each quantity x_i,
  - the only thing we know is the interval $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$ that contains x_i.
- So, the set $\mathbf{y}$ of all possible values of y is the set of all possible values $f(x_1, \ldots, x_n)$ when each x_i is in the corresponding interval:

$$\mathbf{y} = \{f(x_1, \ldots, x_n) : x_i \in \mathbf{x}_i \text{ for all } i\}.$$

- In mathematical terms, the right-hand side of this equality is known as the *range* of the function $f(x_1, \ldots, x_n)$.
- It is usually denoted by $f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.
- For many important classes of problems, there are feasible algorithms which:
  - either compute this range
  - or at least compute a reasonable approximation for this range.

17. Case of interval uncertainty (cont-d)

- The problem of computing this range is known as the problem of *interval computation*.
- It should be mentioned that, in general, the problem of computing this range is NP-hard.
- This means, in effect, that:
  - unless P = NP (which most computer scientists believe to be not true),
  - no feasible algorithm is possible that would *always* compute the exact range.

18. Case of fuzzy uncertainty

- What if for each i and for each x_i, we only know the degree $\mu_i(x_i)$ to which this value x_i is possible?
- In this case, the value y is possible if and only if:
  - for some tuple $(x_1, \ldots, x_n)$ for which $y = f(x_1, \ldots, x_n)$,
  - the value x_1 is possible, *and* the value x_2 is possible, *and* ...
- We know the degree $\mu_i(x_i)$ to which x_i is possible.
- So, to get the degree $\mu(y)$ to which y is possible, we need to apply:
  - an "and"-operation for "and" and
  - an "or"-operation for "for some" (which is nothing else but "or").
- Thus, we get $\mu(y) = f_\vee\{f_\&(\mu_1(x_1), \ldots, \mu_n(x_n)) : y = f(x_1, \ldots, x_n)\}$.
- Here, the "or"-operation is applied to infinitely many values.
- For most "or"-operations, e.g., for $f_\vee(a, b) = a + b - a \cdot b$, if we apply this operation to infinitely many positive terms, we get 1.

19. Case of fuzzy uncertainty (cont-d)

- The only exception is when we use $f_\vee(a,b)=\max(a,b)$.
- In this case, the above expression takes the following form:

$$\mu(y)=\sup_{(x_1,\ldots,x_n):y=f(x_1,\ldots,x_n)} f_\&(\mu_1(x_1),\ldots,\mu_n(x_n)).$$

- This formula was first proposed by Zadeh and is therefore called *Zadeh's extension principle.*
- In particular, for the most commonly used "and"-operation $f_\&(a,b)=\min(a,b)$, we get:

$$\mu(y)=\sup_{(x_1,\ldots,x_n):y=f(x_1,\ldots,x_n)} \min(\mu_1(x_1),\ldots,\mu_n(x_n)).$$

20. These formulas are complex

- The above fuzzy formulas look much more complex than the corresponding interval formulas.
- Interval formulas are, of course, a particular case of the fuzzy formulas:
  - when all fuzzy sets are crisp,
  - i.e., when for each i and each x_i, we have $\mu_i(x_i) = 1$ or $\mu_i(x_i) = 0$.
- This is a known phenomenon – that general computational problems are usually more complex than their particular cases.
- For example:
  - solving systems of linear and quadratic equations is straightforward, we have explicit formulas for these solutions;
  - however, solving general polynomial equations is complicated.

21. These formulas are complex (cont-d)

- Another example:
  - solving systems of linear equations is feasible,
  - however, already solving systems of quadratic equations is NP-hard.
- It was therefore expected that fuzzy computation – i.e., fuzzy data processing – is much more complex than interval computation.
- Lotfi Zadeh himself understood the complexity of this problem.
- He realized that:
  - to make fuzzy methodology practically useful,
  - it is important to develop efficient algorithms for at least some cases of fuzzy computing.

22. Nguyen's theorem: unexpected result

- In 1975, Professor Zadeh:
  - invited Hung T. Nguyen, a promising recent PhD in Mathematics and Statistics from University of Paris,
  - to spend two years at the University of California-Berkeley – to have a mathematician's look at fuzzy theory.
- For this purpose, he asked Hung T. Nguyen to read:
  - his papers and
  - related papers of others – including a paper by a Japanese researcher visiting Berkeley on the computational aspects of fuzzy computing.
- Hung T. Nguyen started working on this topic and came up with a general result about fuzzy computation, published in 1978.
- To explain this result, we need to recall the notion of an α-cut.

23. Nguyen's theorem: unexpected result (cont-d)

- This notion was known in fuzzy methodology because:
  - in many situations, we need to make a decision,
  - e.g., whether to perform a certain action or not.
- E.g., when the satellite deviates a little bit from the desired trajectory, we need to decide:
  - whether we should use the precious fuel to correct its trajectory
  - or not yet.
- If a chemical process starts deviating a little bit from the desired parameter, we need to decide:
  - whether to apply an appropriate control – e.g., shut down the reactor,
  - or not.
- If we know for sure that a sufficiently large deviation took place, then yes, we should perform the corresponding action.

24. Nguyen's theorem: unexpected result (cont-d)

- But what if we can only conclude that this deviation occurred with some degree of confidence d?
- In this case, we need:
  - to select some threshold value $\alpha \in (0, 1]$, and
  - to perform the action if our degree d is larger than or equal to α:

$$d \geq \alpha.$$

- When this degree depends on the value of some quantity x ($d = \mu(x)$), then we perform the action if and only if $\mu(x) \geq \alpha$.
- For each fuzzy set $\mu(x)$, the corresponding set $\{x : \mu(x) \geq \alpha\}$ is known as the α-*cut* of this fuzzy set.

25. Nguyen's theorem: unexpected result (cont-d)

- What Nguyen proved was that:
  - under reasonable conditions,
  - the α-cut $\mathbf{y}(\alpha)$ of y is equal to the range of the function $f(x_1, \ldots, x_n)$ when each x_i is in the α-cut $\mathbf{x}_i(\alpha)$ of x_i:

$$\mathbf{y}(\alpha) = f(\mathbf{x}_1(\alpha), \ldots, \mathbf{x}_n(\alpha)).$$

- The value α corresponds to an expert's degree of confidence in a statement.
- An expert cannot estimate his/her degree with accuracy higher than 0.1.
- Thus, it is sufficient to consider only values α differing by 0.1:

$$0, 0.1, 0.2, \ldots, 1.0.$$

- So, fuzzy computation can reduced to a few cases of interval computation.

26. This theorem is the main tool behind fuzzy computing

- This theorem shows that there is no need to come up with new algorithms for fuzzy computing.
- It is sufficient to use well-developed interval algorithms.
- And this is exactly what most practitioners are doing.
- Every year:
  - there are sessions on interval computations at fuzzy conferences – and
  - there are sessions on possible fuzzy applications at interval conferences.

27. Such situations happen

- The fact that a more general case turned out to be no more computationally complex than a particular case was unexpected.
- However, such situations happened before.
- For example:
  - to come up with equations of General Relativity that describe gravity – i.e., forces caused by masses (= energy),
  - Einstein came up with a completely new idea of curved space-time.
- The general feeling was that:
  - without this new physical idea,
  - we cannot come up with a reasonable explanation for these complex nonlinear partial differential equations.
- However, later, it turned out that the same equations appear if we consider a simple tensor field in flat (not-curved) space time.

28. Such situations happen (cont-d)

- They appear if we make a natural-for-gravity assumption that:
  - the source of this field
  - includes both the energy-momentum of other fields and the energy-momentum of the gravity field itself.
- In physics, there have been many examples of this type, when:
  - a seemingly completely revolutionary theory
  - turned out to be derivable from the previous physics.
- For example:
  - even the notion of the black hole – which was originally perceived as specific for general relativity
  - follows already from Newtonian mechanics.

29. Such situations happen (cont-d)

- Indeed, in Newtonian mechanics:
  - for each celestial body,
  - there is an escape velocity.
- Any object travelling slower than that will fall back to the body.
- If this escape velocity exceeds the speed of light – the largest possible speed – then nothing can leave this body, including light.
- Even the thresholds for when the body with a given mass and radius becomes a black hole are very similar:
  - in General Relativity and
  - in Newton's mechanics.
- From this viewpoint, it is not very surprising that the general fuzzy computing was reduced to a simpler interval computing case.
- Not surprising but still not trivial, since each such reduction requires mathematical and physical ingenuity.

30. Such situations happen (cont-d)

- It took almost 40 years to show that General Relativity can be derived from field theory.
- It took several centuries after Newton to conclude that black holes can exist in Newtonain physics.
- And it took more than 10 years to realize that fuzzy computation can be reduced to interval computations.

31. So how is this theorem proved: main idea

- According to the above formula, the value $\mu(x)$ is the maximum of several values.
- When is the maximum of several numbers larger than or equal to α?
- When one of these numbers if larger than or equal to α: $\mu(y) \ge \alpha \Leftrightarrow$

$$\exists x_1, \ldots, x_n \,(y = f(x_1, \ldots, x_n) = y \,\&\, \min(\mu_1(x_1), \ldots, \mu_n(x_n)) \ge \alpha).$$

- Thus, $y \in \mathbf{y}(\alpha) \Leftrightarrow$

$$\exists x_1, \ldots, x_n \,(y = f(x_1, \ldots, x_n) = y \,\&\, \min(\mu_1(x_1), \ldots, \mu_n(x_n)) \ge \alpha).$$

- When is the smallest of n numbers larger than or equal to α?
- When all of them are larger than or equal to α:

$$\min(\mu_1(x_1), \ldots, \mu_n(x_n)) \ge \alpha \Leftrightarrow \mu_1(x_1) \ge \alpha \,\&\, \ldots \,\&\, \mu_n(x_n) \ge \alpha.$$

- By definition of the α-cut, this means that

$$\min(\mu_1(x_1), \ldots, \mu_n(x_n)) \ge \alpha \Leftrightarrow x_1 \in \mathbf{x}_1(\alpha) \,\&\, \ldots \,\&\, x_n \in \mathbf{x}_n(\alpha).$$

32. So how is this theorem proved: main idea (cont-d)

- Thus:
  - the value y belongs to the α-cut $\mathbf{y}(\alpha)$ if and only if
  - there exist values $x_1, \ldots, x_n$ for which $y = f(x_1, \ldots, x_n)$ and each x_i belongs to the corresponding α-cut $\mathbf{x}_i(\alpha)$.
- In other words, the set $\mathbf{y}(\alpha)$ is indeed equal to the range

$$f(\mathbf{x}_1(\alpha), \ldots, \mathbf{x}_n(\alpha)).$$

- This is exactly what the theorem says.

33. Important warning

- What we described is an idea, but not the full proof.
- It would be a full proof if we had the maximum of finitely many terms.
- However, in our case, we have infinitely many terms.
- It is known that in this case:
  - the supremum may be larger than or equal to α
  - without any of the maximized numbers being larger than or equal to α.
- For example:
  - the supremum of the values $1-2^{-n}$ corresponding to $n = 0, 1, 2 \ldots$ is equal to $\alpha = 1$,
  - while all the values $1 - 2^{-n}$ are smaller than 1.

34. Important warning (cont-d)

- Thus, to have a real proof, we need to guarantee that the supremum is attained for some tuple $(x_1, \ldots, x_n)$.
- This can be guaranteed, e.g., if all the membership functions $\mu_i(x_i)$ are continuous and all the α-cuts are compact.
- For continuous functions of real numbers is equivalent to requiring that all the α-cuts are bounded.
- This requirement which is true for most practical membership functions.

35. Extensions of Nguyen's Theorem beyond real numbers

- The above proof does not depend on the fact that x_i are real numbers:
  - they could be vectors, tuples;
  - they could be, more generally, elements of a general metric (or even general topological) space.
- Such extensions have indeed been published.

36. Extensions to interval-valued, type-2, and more general fuzzy sets

- In the above description, we implicitly assumed that an expert can always describe his/her degree of confidence ny an exact number.
- In reality, however:
  - just like people are not 100% confident about their estimates,
  - they are also not 100% confident about their degrees of confidence.
- A natural idea is to allow:
  - an interval of possible degrees (i.e., use interval-values fuzzy)
  - or even to fuzzy sets describing each degree (i.e., use type-2 fuzzy).
- How to propagate type-2 uncertainty through a data processing algorithm?
- It turned out that such computations can also be reduced to interval computations.

37. Extensions to other t-norms

- In the above text, we considered the case when we use

$$f_{\&}(a, b) = \min(a, b).$$

- What if we use a different t-norm?
- Several extensions of Nguyen's theorem to different t-norms have been proposed.
- It turns out that for other t-norms, we can also have an efficient data processing algorithm.
- Although this time the reduction is not to interval algorithms but to algorithms from convex optimization

38. Other extensions

- An interesting and promising extension was proposed in Fortin et al. 2008, where:
  - the authors represented a fuzzy number – a fuzzy generalization of an interval,
  - as an interval $[\ell(\alpha), r(\alpha)]$ formed by two what they called *gradual numbers* $\ell(\alpha)$ and $r(\alpha)$,
  - i.e., mappings from $(0, 1]$ to the real line.
- The left gradual number $\ell(\alpha)$ is formed by lower endpoints of the α-cut intervals.
- The right gradual number $r(\alpha)$ is formed by its right end-points, so that $[\ell(\alpha), r(\alpha)] = \mathbf{x}(\alpha)$.
- Each of these gradual numbers may not have a clear meaning.
- However, this subdivision seems to simplify computations.

39. Other extensions (cont-d)

- This is just like in physics:
  - while it is not possible to actually separate, e.g., a proton into three quarks,
  - many computations are simplified if we represent a proton this way.
- We hope that other fruitful extensions will occur.

40. References: physics

- R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison Wesley, Boston, Massachusetts, 2005.
- S. N. Gupta, "Quantization of Einstein's gravitational field: general treatment", *Proceedings of Physics Society*, 1952, Vol. A65, pp. 608–619.
- S. N. Gupta, "Gravitation and electromagnetism", *Physical Review*, 1954, Vol. 96, pp. 1683–1685.
- S. N. Gupta, "Einstein's and other theories of gravitation", *Reviews of Modern Physics*, 1957, Vol. 29, pp. 334–336.
- V. Kreinovich, "Gupta's derivation of Einstein equations," *Soviet Physics Doklady*, 1975, Vol. 20, No. 5, pp. 341–342.

41. References: physics (cont-d)

- C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, San Francisco, California, 1973.
- K. S. Thorne and R. D. Blandford, *Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics*, Princeton University Press, Princeton, New Jersey, 2017.

42. References: interval uncertainty

- L. Jaulin, M. Kiefer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control, and Robotics*, Springer, London, 2001.
- V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, *Computational Complexity and Feasibility of Data Processing and Interval Computations*, Kluwer, Dordrecht, 1998.
- G. Mayer, *Interval Analysis and Automatic Result Verification*, de Gruyter, Berlin, 2017.
- R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, 2009.

43. References: fuzzy uncertainty

- R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.
- G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
- J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.
- V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.
- H. T. Nguyen, C. L. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.
- L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.

44. Reference: original Nguyen's Theorem paper

- H. T. Nguyen, "A note on the extension principle for fuzzy sets", *Journal of Mathematical Analysis and Applications*, 1978, Vol. 64, pp. 369–380.

45. References: extensions of Nguyen's Theorem

- C. Carlsson and R. Fuller, "On additions of interactive fuzzy numbers", *Acta Polytechica Hugarica*, 2005, Vol. 2, pp. 59–73.
- L. C. de Barros, R. C. Bassanezi, and P. A. Tonelli, "On the continuity of the Zadeh's extension", *Proceedings of the 7th World Congress of the International Fuzzy Systems Association IFSA'1997*, Prague, June 25–29, 1997, pp. 22–26.
- J. C. de Mota Ferreira, R. C. Bassanezi, and A. J. V. Brandão, *On the Nguyen Theorem for topological spaces*, Preprint, available at https://www.researchgate.net/publication/328538986
- J. Fortin, D. Dubois, and H. Fargier, "Gradual numbers and their application to fuzzy interval analysis", *IEEE Transactions on Fuzzy Systems*, 2008, Vol. 16, No. 2, pp. 388–402.
- R. Fuller and T. Keresztfalvi, "On generalization of Nguyen's theorem", *Fuzzy Sets and Systems*, 1991, Vol. 41, pp. 371–374.

46. References: extensions of Nguyen's Theorem and related papers (cont-d)

- V. Kreinovich, "From processing interval-valued fuzzy data to general type-2: towards fast algorithms", *Proceedings of the IEEE Symposium on Advances in Type-2 Fuzzy Logic Systems T2FUZZ'2011, part of the IEEE Symposium Series on Computational Intelligence*, Paris, France, April 11–15, 2011, pp. ix–xii.

- A. Pownuk, V. Kreinovich, and S. Sriboonchitta, "Fuzzy data processing beyond min t-norm", In: C. Berger-Vachon, A. M. Gil Lafuente, J. Kacprzyk, Y. Kondratenko, J. M. Merigo Lindahl, and C. Morabito (eds.), *Complex Systems: Solutions and Challenges in Economics, Management, and Engineering*, Springer Verlag, 2018, pp. 237–250.

- R. T. Rockafeller, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1997.

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