Why Two Fish Follow Each Other but Three Fish Form a School: A Symmetry-Based Explanation

Shahnaz Shahbazova[1], Olga Kosheleva[2], and Vladik Kreinovich[2]

[1]Department of Digital Technologies and Applied Informatics
Azerbaijan State University of Economics UNEC
Baku, AZ1001, Azerbaijan, shahnaz_shahbazova@unec.edu.az
and Institute of Control Systems
Ministry of Science and Education of the Republic of Azerbaijan
Baku, Azerbaijan, AZ1141, shahbazova@cyber.az

[2]University of Texas at El Paso, 500 W. University
El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu

1. Formulation of the problem

- A recent research analyzed what happens when we place several fish of the same species in an aquarium.
- If there are only two fish, they follow each other.
- If there are three fish, they form a school:
  - they place themselves in positions forming (approximately) an equilateral triangle, and
  - they move in the direction (approximately) orthogonal to this triangle.
- How can we explain this phenomenon?

2. What we do in this talk

- In this talk, we provide a natural symmetry-based explanation for this phenomenon.
- Namely, we show that the observed behavior is optimal:
  - with respect to all optimality criteria
  - that are invariant with respect to natural symmetries: spatial rotations, shifts, and permutations of the fish.
- In this talk:
  - we will not select any specific optimality criterion,
  - we will not specify any numerical model of the fish behavior.
- Instead, we will start with:
  - a kind-of qualitative natural-language description of the situation,
  - namely, the one described above.

3. What we do in this talk (cont-d)

- Then, we will show how this natural-language description can be translated into a precise result.
- In this sense, what we are doing is very similar to what Lotfi Zadeh did when he invented fuzzy techniques.
- Our techniques are different from what Zadeh used.
- However, they still fall under the general rubric of translating natural-language descriptions into precise terms.
- This rubric can be described as – in very general sense – fuzzy.

4. What do we mean by an optimality criterion

- In general, we have the set A of alternatives, and we want to select:
  - the optimal (best) one,
  - i.e., the one which is better than – or of the same quality as – every other alternative.
- Let us denote the relation "better than or of the same quality as" by $\geq$.
- So $a \geq b$ would mean that a is better than b or of the same quality as b.
- In these terms, the optimal alternative $a_{\rm opt}$ is the one for which $a_{\rm opt} \geq a$ for all a.
- Clearly, each alternative is of the same quality as itself, so we have

$$a \geq a.$$

- Relations satisfying this property are known as *reflexive*.

5. What do we mean by an optimality criterion (cont-d)

- Also:
  - if a is better or of the same quality as b, and b is better or of the same quality as c,
  - then a is clearly either better or of the same quality as c.
- Relations satisfying this property are called *transitive.*
- Thus, by an optimality criterion, we will mean a reflexive and transitive relation.
- Such relations are known as *pre-orders*.
- So, we arrive at the following definition.

6. What do we mean by an optimality criterion (cont-d)

- *Let A be a set. Elements of this set will be called* alternatives.
- *By an* optimality criterion *on the set A, we mean a binary relation $\geq$ that satisfies the following two properties:*
  - *for all a, we have $a \geq a$, and*
  - *for all a, b, and c, if $a \geq b$ and $b \geq c$, then $a \geq c$.*
- *We say that the alternative $a_{\rm opt}$ is* optimal *if $a_{\rm opt} \geq a$ for all $a \in A$.*

7. We will only consider final optimality criteria

- Our ultimate goal is to select a single alternative: so:
  - if a current optimality criterion has several equally good optimal alternatives,
  - this means that this criterion is not final: we can use this non-uniqueness to optimize something else.
- For example:
  - if we select the fastest algorithm for solving some problem, and several different algorithms have the same average computation time,
  - we can select, among them, the one that has the smallest worst-case computation time.
- If we still have several equally good algorithms, we can use the remaining non-uniqueness to optimize something else.

8. We will only consider final optimality criteria (cont-d)

- At the end, we should end up with a *final* optimality criterion for which there is exactly one optimal alternative.
- *We say that an optimality criterion is* final *if there is exactly one alternative that is optimal with respect to this criterion.*

9. Natural symmetries

- From the physical viewpoint, there are often some transformations that do not change the relative quality of alternatives.
- For example:
  - if one dish tastes better than another,
  - this relation does not change if we turn and/or shift the table with the taster.
- In general, let G denotes the set of all such transformations.
- If a transformation does not change the relation, the inverse transformation should not change it either.
- If we move a person 100 meters North, then moving the same person 100 meters back South should not affect his/her taste.

10. Natural symmetries (cont-d)

- Similarly, if each of the two transformations does not change the relation:
  - then their composition – when we first apply the first transformation and then the second transformation –
  - also should not change the relation.
- Sets of transformations that contain inverse and composition are known as *transformation groups*.
- So, we arrive at the following definition.

11. Natural symmetries (cont-d)

- *Let A be a set.*
- *By a* transformation group, *we mean a set G of functions $g : A \to A$ for which:*
  - *if the function g is in the set G, then the inverse function g^{-1} exists and is also in the set G;*
  - *if the functions f and g are in the set G, their composition $g(f(a))$ is also in the set G.*
- *We say that the optimality criterion $\geq$ is G-*invariant *if for all $g \in G$ and for all $a, b \in G$, we have $a \geq b$ if and only if $g(a) \geq g(b)$.*
- In physics, invariance is one of the main tools,.
- There, transformations that keep some things invariant are called *symmetries*.
- In line with this, we will also call such transformations symmetries.

12. The result that we will use

- *Let A be a set, and let G be a transformation group on A.*
- *We say that an element $a \in A$ is G-*invariant *if for all $g \in G$, we have*

$$g(a) = a.$$

- **Proposition.** *Let $\geq$ be a final G-invariant optimality criterion $\geq$, then its optimal alternative $a_{\rm opt}$ is G-invariant.*

13. Proof

- To prove this result, we need to prove that, for each $g \in G$, we have

$$g(a_{\text{opt}}) = a_{\text{opt}}.$$

- Indeed, by definition of an optimal alternative, a_{opt} is better than or of the same quality as any other alternative.
- In particular, for each a, we have $a_{\text{opt}} \geq g^{-1}(a)$.
- Since the optimality criterion $\geq$ is G-invariant, we can conclude that

$$g(a_{\text{opt}}) \geq g(g^{-1}(a)).$$

- By the definition of an inverse function, we always have $g(g^{-1}(a)) = a$.
- So we conclude that $g(a_{\text{opt}}) \geq a$ for all $a \in A$.
- By the definition of an optimal alternative, this means that the alternative $g(a_{\text{opt}})$ is optimal.

14. Proof (cont-d)

- But our optimality criterion is final.
- This means that there is only one optimal alternative.
- Thus, the two optimal alternatives a_{opt} and $g(a_{\text{opt}})$ cannot be different, they must be equal: $g(a_{\text{opt}}) = a_{\text{opt}}$.
- The statement is proven.

15. Case of two fish: location and its symmetries

- Whatever two locations the two fish select to place themselves in, these two points form a line.
- One can see that the this 2-point spatial configuration:
  - is not invariant with respect to any shifts,
  - but it is invariant with respect to several rotations.
- We can list all the rotations that keep this spatial configuration invariant:
  - all rotations around the fish-connecting line; these rotations keep both locations intact, and
  - all 180 degree rotations around a different line.
- This is a line which passes through the midpoint between the locations and with is orthogonal to the fish-connecting line.
- Rotations around this line swap the two locations.

16. How to describe possible motions

- At first glance, it seems that:
  - to describe the direction of motion of this spatial configuration,
  - we need to describe the unit vector e in the direction of this motion.
- This would have been true:
  - if we considered a motion with a target destination,
  - e.g., when the fish are pursued by a predator and try to reach a safe zone, which the predator cannot penetrate.
- However, in the experiments that we are trying to explain:
  - we are not talking about a clearly time-directed motion,
  - we are talking about moving in circles.
- In this case, it should not matter whether we consider motions forward in time or the same motions viewed backward in time.

17. How to describe possible motions (cont-d)

- Backward in time simple means that:
  - we reverse the direction of all velocities,
  - i.e., we consider the vector $-e$ instead of the vector e.
- From this viewpoint, what we want to describe is:
  - not so much a unit vector,
  - but rather the direction, the pair $(e, -e)$ consisting of two opposite unit vectors.

18. Which motion is optimal

- A motion is a dynamic configuration consisting of the fish locations and of the motion-related pair $(e, -e)$.
- It is reasonable to assume that the relative quality of different motions should not change under possible rotations.
- Thus, by the above result:
  - the optimal motion
  - should be invariant with respect to all the rotations with respect to which the initial spatial configuration is invariant.
- One can easily see that:
  - if the vector e is not parallel to the fish-connecting line,
  - then the corresponding dynamic configuration is no longer invariant with respect to all the rotations around this line.
- Indeed, each such rotation will change the direction of the vector e.

19. Which motion is optimal (cont-d)

- Thus, the only invariant dynamic configuration is the one:
  - in which the vector e is parallel to the fish-connecting line, i.e.,
  - when fish follow each other.
- We have proved that the optimal motion should lead to an invariant dynamic configuration.
- This means that the optimal motion is exactly the motion in which the fish follow each other.
- This is exactly what was observed.

20. Case of three fish

- To describe the locations of three fish, we need to select three spatial points (x_1, x_2, x_3).
- As we have mentioned, rotations and shifts should not change the relative quality of different spatial locations.
- So, instead of a single triple of points:
  - it make sense to consider, as alternatives,
  - the sets s of all possible locations that are obtained from a triple (x_1, x_2, x_3) by all possible shifts and rotations.

21. Which locations are optimal?

- Which of these sets s is optimal?
- Since we are talking about similar fish:
  - it should not matter which of them we consider fish number 1 and which fish number 2,
  - the relative quality of different spatial configurations should not change.
- In other words, the optimality criterion should be invariant with respect to all possible permutations of fish.
- According to our main result, this means that:
  - the optimal location-describing alternative s
  - should also be invariant under all permutations.

22. Which locations are optimal (cont-d)

- This means, for example, that:
  - if we rename fish 1 and 3,
  - then the resulting triple (x_3, x_2, x_1) should belong to the same optimal set s,
  - i.e., it can be obtained from the original triple (x_1, x_2, x_3) by shifts and rotations.
- Shifts and rotations do not change distance between points.
- So we conclude that:
  - the distance $d(x_3, x_2)$ between the points x_3 and x_2 should be equal to
  - the distance $d(x_1, x_2)$ between the locations of similar fish in the original triple.
- In other words, two sides of the triangle formed by the three fish should be equal.

23. Which locations are optimal (cont-d)

- By considering a different permutation, we can conclude that the third side should also be equal to the other two sides.
- So the optimal spatial configuration should indeed be an equilateral triangle, exactly as observed.

24. What are the symmetries of this spatial configuration

- One can see that the only rotations preserving this spatial configuration are:
  - rotations by 120 and 240 degrees
  - around an axis α which is orthogonal to the plane formed by the fish and which passes through the center of the fish triangle.

25. What are the optimal motions?

- As we have mentioned, in general, a motion can be characterized by a pair $(e, -e)$ of opposite unit vectors.
- According to our main result:
  - the optimal dynamic configuration – consisting of the fish locations and of the motion-describing pair $(e, -e)$,
  - must be invariant with respect to all the corresponding symmetries.
- In our case, invariant with respect to 120- and 240-degree rotations around α.
- One can easily check that:
  - if the vector e is not parallel to α,
  - then the corresponding dynamic configuration is not invariant with respect to such rotations.
- Indeed, its orthogonal-to-α component changes when we rotate.

26. What are the optimal motions (cont-d)

- Thus, the only invariant direction of motion is in the direction of α, i.e., in the direction orthogonal to the fish plane.
- We proved that the optimal direction should be invariant.
- We thus conclude that in the three fish case:
  - the motion corresponding to the optimal dynamic configuration
  - is the motion in the direction orthogonal to the fish triangle.
- This is exactly what was observed.

27. Summarizing

- In both cases, symmetry-based approach to optimization shows that:
  - the optimal spatial configuration and the optimal direction of motion
  - are exactly what was observed in the recent experiments.
- Thus, both observed tendencies can be explained by the fact that fish act optimally.
- This conclusion:
  - does not depend on what exactly optimality criterion the fish use,
  - as long as it is invariant – as it should be – with respect to all possible rotations, shifts, and permutations of fish.

28. Acknowledgments

This work was supported in part by:

- National Science Foundation grants 1623190, HRD-1834620, HRD-2034030, and EAR-2225395;
- AT&T Fellowship in Information Technology;
- program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and
- a grant from the Hungarian National Research, Development and Innovation Office (NRDI).