Interval Estimates: How to Make Them More Adequate and How to Use Them in Economic Analysis and Decision Making

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1. Overview

- In many real-life situations, we need to make decisions under uncertainty; often:
  - instead of the exact values of the relevant quantities,
  - we only know lower and upper bounds on these values.
- In other words, we know an interval that contains the actual (unknown) value.
- These interval estimates often come from experts.
- This fact naturally leads to the following important questions.
- How should we make decisions under such interval uncertainty?
- How to gauge the quality of the resulting decisions? And:
  - if this quality is not sufficient – because the original intervals were too wide,
  - how can we improve the interval estimates so as to make better decisions?

2. Overview (cont-d)

- And if improvements are possible, why not do them from the very beginning, as a pre-processing of expert-provided intervals?
- In this thesis, we propose answers to these questions in several economically meaningful situations.
- We start with a general description of how rational decisions should be made – according to decision theory.
- To make these decisions, we need to have some information about the corresponding quantities.
- This information often comes in terms of expert-provided intervals.
- In Chapter 2, we analyze how these intervals can be improved.
- In Chapter 3, we analyze how we can take interval uncertainty into account when gauging of quality of the existing decisions.
- Finally, in Chapter 4, we analyze how to make new decisions under interval uncertainty.

3. How Economic Decisions Are Made

- We want to provide a numerical value to any actual alternative A.
- We select two alternatives:
  - a very bad alternative A_- which is worse than anything we will actually encounter, and
  - a very good alternative A_+ which is better than anything we will actually encounter.
- For that, we ask the person to compare A with lotteries $L(p)$ in which:
  - this person gets A_+ with some probability p and
  - this person gets A_- with the remaining probability $1 - p$.
- When p is small, $L(p) \leq A$; when p is close to 1, we have $A \leq L(p)$.
- At some probability p_0, the decision maker switches from $L(p) < A$ to $A < L(p)$.
- This p_0 is known as the *utility* $u(A)$ of the alternative A.

4. How Economic Decisions Are Made (cont-d)

- The utility depends on our choice of A_- and A_+.
- If we select a different pair A'_-, A'_+, then the new value $u'(A)$ is related to the original value $u(A)$ by a linear dependence $u'(A) = a \cdot u(A) + b$.
- In the ideal world, a person selects the alternative with the largest utility u_i.
- In reality, the person may select other alternatives as well, with probability $p_i = \dfrac{\exp(\alpha \cdot u_i)}{\sum\limits_{j=1}^{n} \exp(\alpha \cdot u_j)}$.

5. First Result: Preference Relation Is Fundamental in Decision Making

- The preference relation is described by the probabilities p_i.
- We show that they determine utilities uniquely – modulo a linear transformation.
- **Proposition.**
  - *Suppose that for some values* u_i, u'_i, $\alpha > 0$ *and* $\alpha' > 0$, *we have*

$$p_i \stackrel{\text{def}}{=} \frac{\exp(\alpha \cdot u_i)}{\sum\limits_{j=1}^{n} \exp(\alpha \cdot u_j)} = p'_i \stackrel{\text{def}}{=} \frac{\exp(\alpha' \cdot u'_i)}{\sum\limits_{j=1}^{n} \exp(\alpha' \cdot u'_j)}.$$

  - *Then there exists values* $a > 0$ *and* b *for which* $u'_i = a \cdot u_i + b$ *for all* i.

6. Proof of the First Result

- For each $i \neq 1$, if we divide the equality $p_i = p'_i$ by the equality $p_1 = p'_1$, we get $\dfrac{p_i}{p_1} = \dfrac{p'_i}{p'_1}$.
- Thus, $\dfrac{\exp(\alpha \cdot u_i)}{\exp(\alpha \cdot u_1)} = \dfrac{\exp(\alpha' \cdot u'_i)}{\exp(\alpha' \cdot u'_1)}$, i.e., equivalently:

$$\exp(\alpha \cdot (u_i - u_1)) = \exp(\alpha' \cdot (u'_i - u'_1)).$$

- By taking log of both sides and dividing both sides by α', we conclude that $u'_i - u'_1 = a \cdot (u_i - u_1)$, where we denoted $a \stackrel{\text{def}}{=} \alpha/\alpha'$.
- Hence, $u'_i = a \cdot u_i + b$, for $b = u'_1 - a \cdot u_1$. Q.E.D.

7. Interval Estimates and How to Make Them More Adequate

- When an expert provides an interval of possible values:
  - sometimes, the expert is too confident, and the interval provided by the expert is too narrow; we need to widen it;
  - sometimes, the expert is too cautious, and the interval provided by the expert is too wide; we need to make it narrower.
- In such cases, we need to correct these intervals.
- Empirical data shows the corrected version $[A, B]$ of the original interval $[a, b]$ usually follows the formulas

$$A(a,b) = a \cdot \frac{1+\alpha}{2} + b \cdot \frac{1-\alpha}{2}, \quad B(a,b) = a \cdot \frac{1-\alpha}{2} + b \cdot \frac{1+\alpha}{2}.$$

- We provide an explanation for these formulas.

8. Towards Explaining the Empirical Formula

- The same financial predictions can be described in different monetary units, e.g., pesos or dollars.
- If we correct the interval $[a, b]$, we get $[A(a, b), B(a, b)]$.
- Alternatively, we can:
  - first translate into a different monetary unit, getting $[\lambda \cdot a, \lambda \cdot b]$;
  - make a correction there, getting $[A(\lambda \cdot a, \lambda \cdot b), B(\lambda \cdot a, \lambda \cdot b)]$, and
  - translate the result back into the original monetary unit, as

$$\left[\frac{1}{\lambda} \cdot A(\lambda \cdot a, \lambda \cdot b), \frac{1}{\lambda} \cdot B(\lambda \cdot a, \lambda \cdot b)\right].$$

- It is reasonable to require that the corrected interval should be the same whether we use the original monetary units or different units:

$$[A(a, b), B(a, b)] = \left[\frac{1}{\lambda} \cdot A(\lambda \cdot a, \lambda \cdot b), \frac{1}{\lambda} \cdot B(\lambda \cdot a, \lambda \cdot b)\right].$$

- This property is called *scale-invariance*.

9. Shift-Invariance

- Suppose that the expected company's income consists of:
  - the fixed amount f – e.g., determined by the current contracts,
  - and some additional amount x that will depend on the relation between supply and demand.
- Suppose that the expert predicts this additional amount to be somewhere in the interval $[a, b]$.
- This means that the overall company's income is predicted to be between $f+a$ and $f+b$, i.e., somewhere in the interval $[f+a, f+b]$.
- If we believe that the expert estimate needs corrections, then we have two possible ways to perform this correction.
- We can apply the correction to $[a, b]$, resulting in the corrected interval $[A(a, b), B(a, b)]$ for the additional income.
- In this case, the resulting corrected interval for the overall income is

$$[f + A(a, b), f + B(a, b)].$$

10. Shift-Invariance (cont-d)

- Alternatively, we can correct the interval $[a+f, b+f]$ into

$$[A(f+a, f+b), B(f+a, f+b)].$$

- It is reasonable to require that the two methods lead to the exact same interval estimate for the overall income:

$$[f+A(a,b), f+B(a,b)] = [A(f+a, f+b), B(f+a, f+b)].$$

- This is known as *shift-invariance*.

11. Sign Invariance

- One of the possible expert predictions is, e.g., how much bank B_1 will owe a bank B_2 at a certain future date.
- This amount can be positive – meaning that the bank B_1 will owe some money to the bank B_2.
- This amount can also be negative – meaning that the bank B_2 will owe money to the bank B_1.
- Suppose that the expert estimates this amount by an interval $[a, b]$;
  - if we ask the same expert a different question: how much money will the bank B_2 owe to the bank B_1,
  - this expert will provide the interval $[-b, -a]$.
- In this case, we also have two possible ways to correct:
  - we correct $[a, b]$ into $[A(a, b), B(a, b)]$;
  - alternatively, we can correct $[-b, -a]$ and then change the sign, getting $[-B(-b, -a), -A(-b, -a)]$.

12. Sign Invariance (cont-d)

- It is reasonable to require that the two methods should lead to the exact same interval estimate for the overall amount:

$$[A(a,b), B(a,b)] = [-B(-b,-a), -A(-b,-a)].$$

- This property is known as *sign-invariance*.

13. Resulting Explanation

- **Proposition.** *If a mapping* $[a,b]\mapsto[A(a,b),B(a,b)]$ *is scale-, shift-, and sign-invariant, then*

$$A(a,b)=a\cdot\frac{1+\alpha}{2}+b\cdot\frac{1-\alpha}{2},\quad B(a,b)=a\cdot\frac{1-\alpha}{2}+b\cdot\frac{1+\alpha}{2}.$$

- **Proof.** Due to shift-invariance for $f=a$, we have

$$B(a,b)=a+B(0,b-a).$$

- Due to scale-invariance for $\lambda=b-a$, we have

$$B(0,1)=\frac{1}{b-a}\cdot B(0,b-a),\ \text{hence}\ B(0,b-a)=(b-a)\cdot B(0,1).$$

- Let us denote $\alpha\stackrel{\text{def}}{=}2B(0,1)-1$, then $B(0,1)=\dfrac{1+\alpha}{2}$, and the above formula takes the form

$$B(0,b-a)=(b-a)\cdot\frac{1+\alpha}{2}=b\cdot\frac{1+\alpha}{2}-a\cdot\frac{1+\alpha}{2}.$$

14. Proof (cont-d)

- Substituting this expression into the formula for $B(a,b)$, we get

$$B(a,b)=a+b\cdot\frac{1+\alpha}{2}-a\cdot\frac{1+\alpha}{2}=a\cdot\frac{1-\alpha}{2}+b\cdot\frac{1+\alpha}{2}.$$

- This is exactly the desired expression for $B(a,b)$.
- Now, by using sign-invariance, we conclude that

$$A(a,b)=-B(-b,-a)=-\left((-b)\cdot\frac{1-\alpha}{2}+(-a)\cdot\frac{1+\alpha}{2}\right)=$$

$$a\cdot\frac{1+\alpha}{2}+b\cdot\frac{1-\alpha}{2}.$$

- This is also exactly the desired expression. Q.E.D.

15. Economic Analysis under Interval Uncertainty: How Effective Are We?

- In order to make new decisions, we need to gauge the quality of the existing decisions.
- In particular, in many economic applications, it is desirable to estimate how effective is a given country, a given plant, a given farm.
- In some cases, such an estimate is easy.
- For example, if we have a similar farm which is much more productive, then it is clear than the original farm is not effective.
- However, in many situations, such a direct comparison is not possible.
- Difference in countries' productivity may be due to difference in climate, etc., and not necessarily to ineffectiveness.
- To estimate effectiveness in such situations, special Stochastic Frontier techniques have been invented.

16. How Effective Are We (cont-d)

- The current techniques are based on arbitrary assumptions about the probability distribution of effectiveness.
- These assumptions which are motivated mostly by computational efficiency and which do not have any convincing economic motivations.
- Because of this arbitrariness, the conclusions of Stochastic Frontier analysis are often not convincing to users.
- To make these conclusions more convincing, we propose to use economically motivated families of distributions.
- For the new families, the corresponding computational complexity may increase slightly.
- However, the corresponding estimation algorithms are still very efficient.
- These algorithms are based on such actively-used-in-economics techniques as least squares and linear programming.

17. Stochastic Frontier: Main Idea

- We consider objects described by parameters x_i.
- The productivity y of an object is $y = f \cdot r \cdot f(x_1, \ldots, x_n)$, where $f \le 1$ is inefficiency and r is randomness.
- This is usually described in terms of logarithms $Y = \ln(y)$, …, as

$$Y = F + R + F(x_1, \ldots, x_n).$$

- The functions $F(x_1, \ldots, x_n)$ are usually selected from a linear family

$$F(x_1, \ldots, x_n) = G_0(x_1, \ldots, x_n) + \sum_{i=1}^{m} c_i \cdot G_i(x_1, \ldots, x_n).$$

- If we take sufficiently many parameters into account, then $r \approx 1$, so $R \approx 0$, and for each object k, we have

$$Y_k = F_k + G_{k,0} + \sum_{i=1}^{m} c_i \cdot G_{i,k}.$$

18. Stochastic Frontier: Main Idea (cont-d)

- We fix a family of probability distributions for F_k.
- We find the most probable values of c_i and of the parameters of the F_k's distribution.
- This is known as the Maximum Likelihood approach.
- Once we know c_i, we can estimate $F_k = Y_k - G_{k,0} - \sum_{i=1}^{m} c_i \cdot G_{i,k}$ and $f_k = \exp(F_k)$.

19. Which Distributions Are Used Now

- In most applications, there is not enough data to determine this distribution based on the empirical data.
- In practice, usually, the distribution of F_k is selected:
  - either as exponential
  - or as half-Gaussian – i.e., Gaussian with mean 0 limited to negative values of F_k.
- The only reason is that this leads to efficient algorithms.
- The problem with this selection is that it lacks good *economic* motivations.
- Thus, the results of using these distributions are not very convincing.
- This is especially true since these two distributions lead, in general, to different effectiveness results.

20. First Alternative: Normal Distributions

- In most real-life situations, there are many different reasons leading to the decrease in productivity:
  - from not-very-effective upper level management
  - to not-very-effective middle level management all the way
  - to not well trained (thus very effective) workers.
- Each factor $f_{k,i} < 1$ decreases productivity, so $f_k = f_{k,1} \cdot f_{k,2} \cdot \ldots \cdot f_{k,N}$.
- Thus, for logarithms, we have $F_k = F_{k,1} + F_{k,2} + \ldots + F_{k,N}$.
- According to the Central Limit Theorem, the distribution of the sum F_k of many small independent random variables is close to Gaussian.
- So, maximum likelihood means finding c_i that minimize the sum:

$$\sum_{k=1}^{K} \left(Y_k - \mu - G_{k,0} - \sum_{i=1}^{m} c_i \cdot G_{k,i} \right)^2 .$$

- This can be done by the usual Least Squares techniques.

21. Second Alternative: Uniform Distributions

- We do not know anything about the values F_k.
- All we know is that companies, farms, etc. cannot be too ineffective.
- A farm can produce 2, even maybe 10 times less crops than under effective measurement, but not 100 or 1000 times less.
- In other words, all the know is that the values F_k are all located on an interval $[F_0, 0]$.
- We have no reasons to believe that some values from this interval are more probable and some are less probable.
- Thus, it makes sense to conclude that all the values from this interval are equally probable.
- So, we have a uniform distribution on this interval.
- To find the values c_i and F_0, it is reasonable to use the maximum likelihood approach.

22. Second Alternative: Uniform Distributions (cont-d)

- For the uniform distribution, the probability density is equal to $1/|F_0|$.
- The overall probability density is equal to the product of K such terms, i.e., to $1/|F_0|^K$.
- The largest value of this likelihood corresponds to the smallest possible value of $|F_0|$ – i.e., to the largest possible value of $F_0 = -|F_0|$.
- Thus, we need to find the values the largest possible values of F_0 under the constraints:

$$F_0 \le Y_k - G_{k,0} - \sum_{i=1}^{m} c_i \cdot G_{k,i} \le 0, \quad 1 \le k \le K.$$

- This is a particular case of linear programming.
- For linear programming, there are efficient algorithms.

23. How to Make Decisions under Interval Uncertainty

- Sometimes, the only information that we have about a possible choice is that the gain will be in $[\underline{m}, \overline{m}]$.
- Let us denote the price that we are willing to pay for this option by $f(\underline{m}, \overline{m})$.
- Of course, $\underline{m} \le f(\underline{m}, \overline{m}) \le \overline{m}$.
- Also, the function $f(\underline{m}, \overline{m})$ should be monotonic:

$$\text{if } \underline{m} \le \underline{m}' \text{ and } \overline{m} \le \overline{m}', \text{ then } f(\underline{m}, \overline{m}) \le f(\underline{m}', \overline{m}').$$

- The usual recommendation is to have $f(\underline{m}, \overline{m}) = \alpha \cdot \overline{m} + (1 - \alpha) \cdot \underline{m}$.
- This formula was first proposed by a Nobelist Leo Hurwicz.

24. Additivity

- Its usual derivation is based on the following *additivity* assumption.
- Suppose that we have two situations:
  - in the first situation, we can get any value from $\underline{a}$ to $\overline{a}$, and
  - in the second situation, we can get any value from $\underline{b}$ to $\overline{b}$.
- Then, for both situations, we pay $f(\underline{a},\overline{a}) + f\left(\underline{b},\overline{b}\right)$.
- We can consider these two choices as a single situation.
- In this case, the smallest possible gain is $\underline{a}+\underline{b}$, the largest is $\overline{a}+\overline{b}$.
- So, we pay $f\left(\underline{a}+\underline{b},\overline{a}+\overline{b}\right)$.
- It is reasonable to require that these two prices should be equal:

$$f\left(\underline{a}+\underline{b},\overline{a}+\overline{b}\right) = f(\underline{a},\overline{a}) + f\left(\underline{b},\overline{b}\right).$$

- This property is called *additivity.*

25. Additivity Is Not Fully Convincing

- Additivity means that:
  - if the worst-case scenario is possible for each of the two situations,
  - then it is possible that we have the worst-case scenario in both situations.
- This is not full agreement with common sense.
- When we fly from point A to point B, we understand:
  - that there may an unexpected delay at the airport A,
  - that a plane may have a problem in flight and we will have to get back, etc.
- However, we honestly do *not* believe that all these low-probable disasters will happen at the same.
- This only happens in comedies describing lovable losers who always get into trouble.

26. Additivity Is Not Fully Convincing (cont-d)

- Another assumption is that for the combination of two items, we always pay the same price as for the two items separately.
- This is not always true.
- There often *are* discounts if you buy several items.

27. Alternatives to Additivity

- Suppose that we offer a user a package deal in which he/she gets:
  - m dollars cash *and*
  - an alternative in which he/she gets between $\underline{m}$ and $\overline{m}$.
- The equivalent value for the interval-value alternative is $f(\underline{m}, \overline{m})$.
- So the overall value for this package is $m + f(\underline{m}, \overline{m})$.
- On the other hand, in this deal, we get any amount between $m + \underline{m}$ and $m + \overline{m}$.
- So, the value of this package deal should be $f(m + \underline{m}, m + \overline{m})$.
- It is reasonable to require that these two valuations should lead to the same result: $m + f(\underline{m}, \overline{m}) = f(m + \underline{m}, m + \overline{m})$.
- This property is called *shift-invariance.*

28. Alternatives to Additivity (cont-d)

- Shift-invariance is not enough: we can have $f(\underline{m}, \overline{m}) = \underline{m} + F(\overline{m} - \underline{m})$, for any monotonic function $F(z)$ for which $F(z) \le z$.
- So, we need an additional requirement.
- Let us denote $\alpha \stackrel{\text{def}}{=} f(0,1)$. Due to shift invariance, we for every $x \in [0,1]$, we have $f(x, 1+x) = \alpha + x$, i.e.: $[x, 1+x] \equiv \alpha + x$.
- The union of intervals $[x, 1+x]$ is $[0,2]$.
- Each of the united intervals is equivalent to the value $\alpha + x$.
- Thus, the union of these intervals is equivalent to the set of all possible values $\alpha + x$ when $x \in [0,1]$:

$$[0,2] = \bigcup_{x \in [0,1]} [x, 1+x] \equiv \{\alpha + x : x \in [0,1]\}.$$

- The right-hand side of this equality is the interval $[\alpha, 1+\alpha]$ whose price is 2α; thus, $f(0,2) = 2\alpha$.

29. Result

- We say that $f(\underline{m}, \overline{m})$ is *reasonable* if it is shift-invariant and for all $\underline{m}$, $\overline{m}$, and w, we have $f\left(\underline{\ell}, \overline{\ell}\,\right) = f(\underline{r}, \overline{r})$, where

$$\left[\underline{\ell}, \overline{\ell}\,\right] \stackrel{\text{def}}{=} \bigcup_{m \in [\underline{m}, \overline{m}]} [m, w + m] \text{ and } [\underline{r}, \overline{r}] \stackrel{\text{def}}{=} \{f(m, m + w) : m \in [\underline{m}, \overline{m}]\}.$$

- **Proposition.** *Every reasonable function has a Hurwicz form.*