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1. Overview

- In many real-life situations, we need to make decisions under uncertainty; often:
  - instead of the exact values of the relevant quantities,
  - we only know lower and upper bounds on these values.
- In other words, we know an interval that contains the actual (unknown) value.
- These interval estimates often come from experts.
- This fact naturally leads to the following important questions.
- How should we make decisions under such interval uncertainty?
- How to gauge the quality of the resulting decisions? And:
  - if this quality is not sufficient – because the original intervals were too wide,
  - how can we improve the interval estimates so as to make better decisions?
2. Overview (cont-d)

- And if improvements are possible, why not do them from the very beginning, as a pre-processing of expert-provided intervals?
- In this thesis, we propose answers to these questions in several economically meaningful situations.
- We start with a general description of how rational decisions should be made – according to decision theory.
- To make these decisions, we need to have some information about the corresponding quantities.
- This information often comes in terms of expert-provided intervals.
- In Chapter 2, we analyze how these intervals can be improved.
- In Chapter 3, we analyze how we can take interval uncertainty into account when gauging of quality of the existing decisions.
- Finally, in Chapter 4, we analyze how to make new decisions under interval uncertainty.
3. How Economic Decisions Are Made

- We want to provide a numerical value to any actual alternative $A$.
- We select two alternatives:
  - a very bad alternative $A_-$ which is worse than anything we will actually encounter, and
  - a very good alternative $A_+$ which is better than anything we will actually encounter.
- For that, we ask the person to compare $A$ with lotteries $L(p)$ in which:
  - this person gets $A_+$ with some probability $p$ and
  - this person gets $A_-$ with the remaining probability $1 - p$.
- When $p$ is small, $L(p) \leq A$; when $p$ is close to 1, we have $A \leq L(p)$.
- At some probability $p_0$, the decision maker switches from $L(p) < A$ to $A < L(p)$.
- This $p_0$ is known as the utility $u(A)$ of the alternative $A$. 
4. How Economic Decisions Are Made (cont-d)

- The utility depends on our choice of $A_-$ and $A_+$. 
- If we select a different pair $A'_-$, $A'_+$, then the new value $u'(A)$ is related to the original value $u(A)$ by a linear dependence $u'(A) = a \cdot u(A) + b$. 
- In the ideal world, a person selects the alternative with the largest utility $u_i$. 
- In reality, the person may select other alternatives as well, with probability $p_i = \frac{\exp(\alpha \cdot u_i)}{\sum_{j=1}^{n} \exp(\alpha \cdot u_j)}$. 
5. First Result: Preference Relation Is Fundamental in Decision Making

- The preference relation is described by the probabilities $p_i$.
- We show that they determine utilities uniquely – modulo a linear transformation.
- Proposition.

  - Suppose that for some values $u_i$, $u'_i$, $\alpha > 0$ and $\alpha' > 0$, we have
    
    \[ p_i \overset{\text{def}}{=} \frac{\exp(\alpha \cdot u_i)}{\sum_{j=1}^{n} \exp(\alpha \cdot u_j)} = p'_i \overset{\text{def}}{=} \frac{\exp(\alpha' \cdot u'_i)}{\sum_{j=1}^{n} \exp(\alpha' \cdot u'_j)}. \]

  - Then there exists values $a > 0$ and $b$ for which $u'_i = a \cdot u_i + b$ for all $i$. 

6. Proof of the First Result

- For each $i \neq 1$, if we divide the equality $p_i = p'_i$ by the equality $p_1 = p'_1$, we get $\frac{p_i}{p_1} = \frac{p'_i}{p'_1}$.

- Thus, $\frac{\exp(\alpha \cdot u_i)}{\exp(\alpha \cdot u_1)} = \frac{\exp(\alpha' \cdot u'_i)}{\exp(\alpha' \cdot u'_1)}$, i.e., equivalently:

  $$\exp(\alpha \cdot (u_i - u_1)) = \exp(\alpha' \cdot (u'_i - u'_1)).$$

- By taking log of both sides and dividing both sides by $\alpha'$, we conclude that $u'_i - u'_1 = a \cdot (u_i - u_1)$, where we denoted $a \overset{\text{def}}{=} \frac{\alpha}{\alpha'}$.

- Hence, $u'_i = a \cdot u_i + b$, for $b = u'_1 - a \cdot u_1$. Q.E.D.
7. Interval Estimates and How to Make Them More Adequate

- When an expert provides an interval of possible values:
  - sometimes, the expert is too confident, and the interval provided by the expert is too narrow; we need to widen it;
  - sometimes, the expert is too cautious, and the interval provided by the expert is too wide; we need to make it narrower.
- In such cases, we need to correct these intervals.
- Empirical data shows the corrected version $[A, B]$ of the original interval $[a, b]$ usually follows the formulas
  \[
  A(a, b) = a \cdot \frac{1 + \alpha}{2} + b \cdot \frac{1 - \alpha}{2}, \quad B(a, b) = a \cdot \frac{1 - \alpha}{2} + b \cdot \frac{1 + \alpha}{2}.
  \]
- We provide an explanation for these formulas.
8. Towards Explaining the Empirical Formula

- The same financial predictions can be described in different monetary units, e.g., pesos or dollars.
- If we correct the interval \([a, b]\), we get \([A(a, b), B(a, b)]\).
- Alternatively, we can:
  - first translate into a different monetary unit, getting \([\lambda \cdot a, \lambda \cdot b]\);
  - make a correction there, getting \([A(\lambda \cdot a, \lambda \cdot b), B(\lambda \cdot a, \lambda \cdot b)]\), and
  - translate the result back into the original monetary unit, as
    \[
    \left[ \frac{1}{\lambda} \cdot A(\lambda \cdot a, \lambda \cdot b), \frac{1}{\lambda} \cdot B(\lambda \cdot a, \lambda \cdot b) \right].
    \]
- It is reasonable to require that the corrected interval should be the same whether we use the original monetary units or different units:
  \[
  [A(a, b), B(a, b)] = \left[ \frac{1}{\lambda} \cdot A(\lambda \cdot a, \lambda \cdot b), \frac{1}{\lambda} \cdot B(\lambda \cdot a, \lambda \cdot b) \right].
  \]
- This property is called *scale-invariance*. 
9. Shift-Invariance

- Suppose that the expected company’s income consists of:
  - the fixed amount $f$ – e.g., determined by the current contracts,
  - and some additional amount $x$ that will depend on the relation between supply and demand.
- Suppose that the expert predicts this additional amount to be somewhere in the interval $[a, b]$.
- This means that the overall company’s income is predicted to be between $f + a$ and $f + b$, i.e., somewhere in the interval $[f + a, f + b]$.
- If we believe that the expert estimate needs corrections, then we have two possible ways to perform this correction.
- We can apply the correction to $[a, b]$, resulting in the corrected interval $[A(a, b), B(a, b)]$ for the additional income.
- In this case, the resulting corrected interval for the overall income is $[f + A(a, b), f + B(a, b)]$. 
10. Shift-Invariance (cont-d)

- Alternatively, we can correct the interval $[a + f, b + f]$ into
  
  $$[A(f + a, f + b), B(f + a, f + b)].$$

- It is reasonable to require that the two methods lead to the exact same interval estimate for the overall income:

  $$[f + A(a, b), f + B(a, b)] = [A(f + a, f + b), B(f + a, f + b)].$$

- This is known as shift-invariance.
11. Sign Invariance

- One of the possible expert predictions is, e.g., how much bank $B_1$ will owe a bank $B_2$ at a certain future date.
- This amount can be positive – meaning that the bank $B_1$ will owe some money to the bank $B_2$.
- This amount can also be negative – meaning that the bank $B_2$ will owe money to the bank $B_1$.
- Suppose that the expert estimates this amount by an interval $[a, b]$;
  - if we ask the same expert a different question: how much money will the bank $B_2$ owe to the bank $B_1$,
  - this expert will provide the interval $[-b, -a]$.
- In this case, we also have two possible ways to correct:
  - we correct $[a, b]$ into $[A(a, b), B(a, b)]$;
  - alternatively, we can correct $[-b, -a]$ and then change the sign, getting $[-B(-b, -a), -A(-b, -a)]$. 
12. Sign Invariance (cont-d)

- It is reasonable to require that the two methods should lead to the exact same interval estimate for the overall amount:

\[ [A(a, b), B(a, b)] = [-B(-b, -a), -A(-b, -a)]. \]

- This property is known as *sign-invariance*. 
13. Resulting Explanation

- **Proposition.** If a mapping \([a, b] \mapsto [A(a, b), B(a, b)]\) is scale-, shift-, and sign-invariant, then

\[
A(a, b) = a \cdot \frac{1 + \alpha}{2} + b \cdot \frac{1 - \alpha}{2}, \quad B(a, b) = a \cdot \frac{1 - \alpha}{2} + b \cdot \frac{1 + \alpha}{2}.
\]

- **Proof.** Due to shift-invariance for \(f = a\), we have

\[
B(a, b) = a + B(0, b - a).
\]

Due to scale-invariance for \(\lambda = b - a\), we have

\[
B(0, 1) = \frac{1}{b - a} \cdot B(0, b - a), \text{ hence } B(0, b - a) = (b - a) \cdot B(0, 1).
\]

Let us denote \(\alpha \overset{\text{def}}{=} 2B(0, 1) - 1\), then \(B(0, 1) = \frac{1 + \alpha}{2}\), and the above formula takes the form

\[
B(0, b - a) = (b - a) \cdot \frac{1 + \alpha}{2} = b \cdot \frac{1 + \alpha}{2} - a \cdot \frac{1 + \alpha}{2}.
\]
14. **Proof (cont-d)**

- Substituting this expression into the formula for $B(a, b)$, we get

$$B(a, b) = a + b \cdot \frac{1 + \alpha}{2} - a \cdot \frac{1 + \alpha}{2} = a \cdot \frac{1 - \alpha}{2} + b \cdot \frac{1 + \alpha}{2}.$$  

- This is exactly the desired expression for $B(a, b)$.

- Now, by using sign-invariance, we conclude that

$$A(a, b) = -B(-b, -a) = - \left( (-b) \cdot \frac{1 - \alpha}{2} + (-a) \cdot \frac{1 + \alpha}{2} \right) =$$

$$a \cdot \frac{1 + \alpha}{2} + b \cdot \frac{1 - \alpha}{2}.$$  

- This is also exactly the desired expression. Q.E.D.
15. Economic Analysis under Interval Uncertainty: How Effective Are We?

- In order to make new decisions, we need to gauge the quality of the existing decisions.
- In particular, in many economic applications, it is desirable to estimate how effective is a given country, a given plant, a given farm.
- In some cases, such an estimate is easy.
- For example, if we have a similar farm which is much more productive, then it is clear than the original farm is not effective.
- However, in many situations, such a direct comparison is not possible.
- Difference in countries’ productivity may be due to difference in climate, etc., and not necessarily to ineffectiveness.
- To estimate effectiveness in such situations, special Stochastic Frontier techniques have been invented.
16. How Effective Are We (cont-d)

- The current techniques are based on arbitrary assumptions about the probability distribution of effectiveness.
- These assumptions which are motivated mostly by computational efficiency and which do not have any convincing economic motivations.
- Because of this arbitrariness, the conclusions of Stochastic Frontier analysis are often not convincing to users.
- To make these conclusions more convincing, we propose to use economically motivated families of distributions.
- For the new families, the corresponding computational complexity may increase slightly.
- However, the corresponding estimation algorithms are still very efficient.
- These algorithms are based on such actively-used-in-economics techniques as least squares and linear programming.
17. Stochastic Frontier: Main Idea

- We consider objects described by parameters $x_i$.
- The productivity $y$ of an object is $y = f \cdot r \cdot f(x_1, \ldots, x_n)$, where $f \leq 1$ is inefficiency and $r$ is randomness.
- This is usually described in terms of logarithms $Y = \ln(y)$, $\ldots$, as $Y = F + R + F(x_1, \ldots, x_n)$.
- The functions $F(x_1, \ldots, x_n)$ are usually selected from a linear family $F(x_1, \ldots, x_n) = G_0(x_1, \ldots, x_n) + \sum_{i=1}^{m} c_i \cdot G_i(x_1, \ldots, x_n)$.
- If we take sufficiently many parameters into account, then $r \approx 1$, so $R \approx 0$, and for each object $k$, we have $Y_k = F_k + G_{k,0} + \sum_{i=1}^{m} c_i \cdot G_{i,k}$. 
18. Stochastic Frontier: Main Idea (cont-d)

- We fix a family of probability distributions for $F_k$.

- We find the most probable values of $c_i$ and of the parameters of the $F_k$’s distribution.

- This is known as the Maximum Likelihood approach.

- Once we know $c_i$, we can estimate $F_k = Y_k - G_{k,0} - \sum_{i=1}^{m} c_i \cdot G_{i,k}$ and $f_k = \exp(F_k)$. 
19. Which Distributions Are Used Now

- In most applications, there is not enough data to determine this distribution based on the empirical data.
- In practice, usually, the distribution of $F_k$ is selected:
  - either as exponential
  - or as half-Gaussian – i.e., Gaussian with mean 0 limited to negative values of $F_k$.
- The only reason is that this leads to efficient algorithms.
- The problem with this selection is that it lacks good economic motivations.
- Thus, the results of using these distributions are not very convincing.
- This is especially true since these two distributions lead, in general, to different effectiveness results.
20. First Alternative: Normal Distributions

- In most real-life situations, there are many different reasons leading to the decrease in productivity:
  - from not-very-effective upper level management
  - to not-very-effective middle level management all the way
  - to not well trained (thus very effective) workers.
- Each factor $f_{k,i} < 1$ decreases productivity, so $f_k = f_{k,1} \cdot f_{k,2} \cdot \ldots \cdot f_{k,N}$.
- Thus, for logarithms, we have $F_k = F_{k,1} + F_{k,2} + \ldots + F_{k,N}$.
- According to the Central Limit Theorem, the distribution of the sum $F_k$ of many small independent random variables is close to Gaussian.
- So, maximum likelihood means finding $c_i$ that minimize the sum:
  \[
  \sum_{k=1}^{K} \left( Y_k - \mu - G_{k,0} - \sum_{i=1}^{m} c_i \cdot G_{k,i} \right)^2.
  \]
- This can be done by the usual Least Squares techniques.
21. Second Alternative: Uniform Distributions

- We do not know anything about the values $F_k$.
- All we know is that companies, farms, etc. cannot be too ineffective.
- A farm can produce 2, even maybe 10 times less crops than under effective measurement, but not 100 or 1000 times less.
- In other words, all we know is that the values $F_k$ are all located on an interval $[F_0, 0]$.
- We have no reasons to believe that some values from this interval are more probable and some are less probable.
- Thus, it makes sense to conclude that all the values from this interval are equally probable.
- So, we have a uniform distribution on this interval.
- To find the values $c_i$ and $F_0$, it is reasonable to use the maximum likelihood approach.
22. Second Alternative: Uniform Distributions (cont-d)

- For the uniform distribution, the probability density is equal to $1/|F_0|$.
- The overall probability density is equal to the product of $K$ such terms, i.e., to $1/|F_0|^K$.
- The largest value of this likelihood corresponds to the smallest possible value of $|F_0|$ – i.e., to the largest possible value of $F_0 = -|F_0|$.
- Thus, we need to find the values the largest possible values of $F_0$ under the constraints:
  \[
  F_0 \leq Y_k - G_{k,0} - \sum_{i=1}^{m} c_i \cdot G_{k,i} \leq 0, \quad 1 \leq k \leq K.
  \]
- This is a particular case of linear programming.
- For linear programming, there are efficient algorithms.
How to Make Decisions under Interval Uncertainty

- Sometimes, the only information that we have about a possible choice is that the gain will be in $[m, \overline{m}]$.
- Let us denote the price that we are willing to pay for this option by $f(m, \overline{m})$.
- Of course, $m \leq f(m, \overline{m}) \leq \overline{m}$.
- Also, the function $f(m, \overline{m})$ should be monotonic:
  
  \[
  \text{if } m \leq m' \text{ and } \overline{m} \leq \overline{m}', \text{ then } f(m, \overline{m}) \leq f(m', \overline{m}').
  \]
- The usual recommendation is to have $f(m, \overline{m}) = \alpha \cdot \overline{m} + (1 - \alpha) \cdot m$.
- This formula was first proposed by a Nobelist Leo Hurwicz.
24. Additivity

- Its usual derivation is based on the following *additivity* assumption.
- Suppose that we have two situations:
  - in the first situation, we can get any value from $a$ to $\bar{a}$, and
  - in the second situation, we can get any value from $b$ to $\bar{b}$.
- Then, for both situations, we pay $f(a, \bar{a}) + f(\bar{b}, b)$.
- We can consider these two choices as a single situation.
- In this case, the smallest possible gain is $a + b$, the largest is $\bar{a} + \bar{b}$.
- So, we pay $f(a + b, \bar{a} + \bar{b})$.
- It is reasonable to require that these two prices should be equal:
  \[ f(a + b, \bar{a} + \bar{b}) = f(a, \bar{a}) + f(\bar{b}, b) \, . \]
- This property is called *additivity*. 
25. Additivity Is Not Fully Convincing

- Additivity means that:
  - if the worst-case scenario is possible for each of the two situations,
  - then it is possible that we have the worst-case scenario in both situations.

- This is not full agreement with common sense.

- When we fly from point A to point B, we understand:
  - that there may an unexpected delay at the airport A,
  - that a plane may have a problem in flight and we will have to get back, etc.

- However, we honestly do not believe that all these low-probable disasters will happen at the same.

- This only happens in comedies describing lovable losers who always get into trouble.
26. Additivity Is Not Fully Convincing (cont-d)

- Another assumption is that for the combination of two items, we always pay the same price as for the two items separately.
- This is not always true.
- There often are discounts if you buy several items.
27. Alternatives to Additivity

- Suppose that we offer a user a package deal in which he/she gets:
  - \( m \) dollars cash and
  - an alternative in which he/she gets between \( m \) and \( \overline{m} \).
- The equivalent value for the interval-value alternative is \( f(m, \overline{m}) \).
- So the overall value for this package is \( m + f(m, \overline{m}) \).
- On the other hand, in this deal, we get any amount between \( m + m \) and \( m + \overline{m} \).
- So, the value of this package deal should be \( f(m + m, m + \overline{m}) \).
- It is reasonable to require that these two valuations should lead to the same result: \( m + f(m, \overline{m}) = f(m + m, m + \overline{m}) \).
- This property is called \textit{shift-invariance}. 
28. Alternatives to Additivity (cont-d)

- Shift-invariance is not enough: we can have \( f(m, m) = m + F(m - m) \), for any monotonic function \( F(z) \) for which \( F(z) \leq z \).

- So, we need an additional requirement.

- Let us denote \( \alpha \overset{\text{def}}{=} f(0, 1) \). Due to shift invariance, we for every \( x \in [0, 1] \), we have \( f(x, 1 + x) = \alpha + x \), i.e.: \( [x, 1 + x] \equiv \alpha + x \).

- The union of intervals \( [x, 1 + x] \) is \([0, 2]\).

- Each of the united intervals is equivalent to the value \( \alpha + x \).

- Thus, the union of these intervals is equivalent to the set of all possible values \( \alpha + x \) when \( x \in [0, 1] \):

\[
[0, 2] = \bigcup_{x \in [0, 1]} [x, 1 + x] \equiv \{ \alpha + x : x \in [0, 1] \}.
\]

- The right-hand side of this equality is the interval \([\alpha, 1 + \alpha]\) whose price is \(2\alpha\); thus, \( f(0, 2) = 2\alpha \).
29. Result

- We say that \( f(\underline{m}, \overline{m}) \) is reasonable if it is shift-invariant and for all \( \underline{m}, \overline{m}, \) and \( w, \) we have \( f(\ell, \ell) = f(r, \overline{r}) \), where

\[
[\ell, \ell] \overset{\text{def}}{=} \bigcup_{m \in [\underline{m}, \overline{m}]} [m, w + m] \text{ and } [r, \overline{r}] \overset{\text{def}}{=} \{ f(m, m + w) : m \in [\underline{m}, \overline{m}] \}.
\]

- Proposition. Every reasonable function has a Hurwicz form.