# Why Beta Priors: Invariance-Based Explanation

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#### 1. Formulation of the Problem

- In the Bayesian approach:
  - when we do not know the probability  $p \in [0, 1]$  of some event,
  - it is usually recommended to use a Beta prior distribution for p, with pdf

$$\rho(x) = c \cdot x^{\alpha - 1} \cdot (1 - x)^{\beta - 1}.$$

- There have been numerous successful application of the use of the Beta distribution in the Bayesian approach.
- How can we explain this success?
- Why not use some other family of distributions located on the interval [0, 1]?



#### 2. Formulation of the Problem (cont-d)

- The need for such an explanation is especially important now, when the statistician community is:
  - replacing the traditional p-value techniques
  - with more reliable hypothesis testing methods.
- One such method is the Minimum Bayesian Factor (MBF) method.
- This method is based on Beta priors  $\rho(x) = c \cdot x^a$  corresponding to  $\beta = 1$ .
- In this paper, we provide a natural explanation for these empirical successes.



#### 3. Main Idea

- We want to find a natural prior distribution on the interval [0, 1].
- ullet This distribution should describe how frequently different probability values p appear.
- In determining this distribution, a natural idea to take into account is that:
  - in practice,
  - all probabilities are, in effect, conditional probabilities.
- We start with some class, and in this class, we find the corresponding frequencies.



#### 4. Main Idea (cont-d)

- From this viewpoint:
  - we can start with the original probabilities and with their prior distribution,
  - or we can impose additional conditions and consider the resulting conditional probabilities.
- For example, in medical data processing, we may consider the probability that:
  - a patient with a certain disease
  - recovers after taking the corresponding medicine.
- We can consider this original probability.
- Alternatively, we can consider the conditional probability that a patient will recover.
- For example, the condition can be that the patient is at least 18 years old.



#### 5. Main Idea (cont-d)

- We can impose many such conditions.
- We are looking for a universal prior, a prior that would describe all possible situations.
- So, it makes sense to consider priors for which:
  - after such a restriction,
  - we will get the exact same prior for the corresponding conditional probability.



#### 6. Let Us Describe This Main Idea in Precise Terms

• In general, the conditional probability  $P(A \mid B)$  has the form

$$P(A \mid B) = \frac{P(A \& B)}{P(B)}.$$

- Crudely speaking, this means that:
  - when we transition from the original probabilities to the new conditional ones,
  - we limit ourselves to the original probabilities which do not exceed some value  $p_0 = P(B)$ , and
  - we divide each original probability by  $p_0$ .
- In these terms, the above requirement takes the following form: for each  $p_0 \in (0,1)$ ,
  - if we limit ourselves to the interval  $[0, p_0]$ ,
  - then the ratios  $p/p_0$  should have the same distribution as the original one.

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## 7. Resulting Definition

- Let us assume that we have a probability distribution with probability density  $\rho(x)$  on the interval [0,1].
- We say that this distribution is invariant if:
  - $for each p_0 \in (0, 1),$
  - the ratio  $x/p_0$  (restricted to the values  $x \leq p_0$ ) has the same distribution, i.e.:

$$\rho(x/p_0: x \le p_0) = \rho(x).$$



$$\rho(x) = c \cdot x^a$$
 for some  $c$  and  $a$ .

• **Proof.** The conditional probability density has the form

$$\rho(x/p_0: x \le p_0) = C(p_0) \cdot \rho(x/p_0).$$

- Here, C is an appropriate constant depending on  $p_0$ .
- Thus, the invariance condition has the form

$$C(p_0) \cdot \rho(x/p_0) = \rho(x).$$

• By moving the term  $C(p_0)$  to the right-hand side and denoting  $\lambda \stackrel{\text{def}}{=} 1/p_0$  (so that  $p_0 = 1/\lambda$ ), we get

$$\rho(\lambda \cdot x) = c(\lambda) \cdot \rho(x).$$

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## 9. Proof (cont-d)

• We get  $\rho(\lambda \cdot x) = c(\lambda) \cdot \rho(x)$ , where we denoted

$$c(\lambda) \stackrel{\text{def}}{=} 1/C(1/\lambda).$$

- The probability density function is an integrable function its integral is equal to 1.
- Known: all integrable solutions of the above functional equation has the form  $\rho(x) = c \cdot x^a$  for some c, a.
- The proposition is thus proven.
- Reminder: these distributions corr. to  $\beta = 1$  are used in the Bayesian approach to hypothesis testing.



#### 10. How to Get a General Prior Distribution

- The above proposition describes the case when:
  - we have a single distribution
  - corresponding to a single piece of prior information.
- In practice, we may have many different pieces of information:
  - some of these pieces are about the probability p of the corresponding event E,
  - some may be about the probability p' = 1 p of the opposite event  $\neg E$ .
- According to the above Proposition, each piece of information about p can be described by the pdf  $c_i \cdot x^{a_i}$ .
- Similarly, each piece of information about p' = 1 p can be described by the probability density

$$c'_j \cdot x^{a'_j}$$
.

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# 11. General Case (cont-d)

• In terms of the original probability p = 1 - p', this probability density has the form

$$c_i' \cdot (1-x)^{a_j'}.$$

- All these piece of information are independent.
- So, a reasonable idea is to multiply these probability density functions.
- After multiplication, we get a distribution of the type  $c \cdot x^a \cdot (a-x)^{a'}$ , where  $a = \sum_i a_i$  and  $a' = \sum_i a'_i$ .
- This is exactly the Beta distribution for  $\alpha = a + 1$  and  $\beta = a' + 1$ .
- $\bullet$  Thus, we have indeed justified the use of Beta priors.

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Formulation of the . . .

Main Idea

Let Us Describe This . . .

Resulting Definition

Main Result and Its Proof

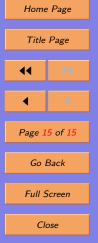
Proof (cont-d)

How to Get a General . . .

Acknowledgments

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