# How to Relate Spectral Risk Measures and Utilities

Songsak Sriboonchitta[1]
Hung T. Nguyen[1,2], and Vladik Kreinovich[1,3]

[1]Faculty of Economics, Chiang Mai University
Chiang Mai 50200 Thailand, songsak@econ.cmu.ac.th

[2]Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003, USA, hunguyen@nmsu.edu

[3]Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA, vladik@utep.edu

# 1. Decision Theory: Traditional Approach

- *Decision theory – main objective:*
  - to formally *describe* how people make decisions, what is preferable and what is not, so as to be able
  - to *help* decision makers by prompting them decisions which should be beneficial for them.
- *Traditional decision theory approach* – in terms of utility functions:
  - We assign, to each alternative $x$, a "utility" value $u(x)$ describing how valuable is $x$.
  - The quality of each action is characterized by the expected values of the corresponding utility.
  - We therefore select an action which leads to the largest value of expected utility.

# 2. Spectral Risk Measures Approach: A Brief Reminder

- *Spectral risk measures approach:*
  - we select a "weighting" function $\varphi(p)$ which assigns a weight to all possible probability values $p \in [0,1]$;
  - we characterize the quality of an action by the value
$$\int_0^1 \varphi(p) \cdot F^{-1}(p)\, dp.$$
- *Comment:* $x_0 = F^{-1}(p)$ is the value for which
$$F(x_0) = p, \text{ i.e. } \mathrm{Prob}(x \le x_0) = p.$$
- *Fact:* both utility and spectral risk measures provide a reasonable description of the same economic behavior.
- *Conclusion:* there should be a relationship between:
  - utility function (from the traditional approach) and
  - the weighting function (from spectral risk measures).

# 3. Heuristic Relation Between Utility and Weighting Functions

- *Fact:* there are empirical rules relating utility functions $u(x)$ and weighting functions $\varphi(p)$; e.g.:
  - to the exp $u(x) = 1 - e^{-k\cdot x}$, the exponential f-n:

$$\varphi(p) = \frac{k}{1 - e^{-k}} \cdot \exp(-k \cdot (1 - p));$$

  - to the power $u(x) = x^{1-\gamma}$, the power weighting f-n:

$$\varphi(p) = \gamma \cdot (1 - p)^{\gamma - 1}.$$

- *Problems:*
  - these rules are purely heuristic;
  - the proposed match is not perfect: e.g., behaviors for power $u(x)$ and power $\varphi(p)$ are different.
- *Desirable:* provide a theoretically justified relation.

# 4. Our Idea: Use Statistics

- *Fact:* both approaches can be reformulated in statistical terms.
- *Idea:* use these reformulations to provide a statistically justified relation between $u(x)$ and $\varphi(p)$.
- *Utility theory:* fix $A_0 < A_1$ and define $u(x)$ as the probability for which $x \sim$ "$A_1$ w/prob. $p$ and $A_0$ otherwise".
- *Known:* changing $A_i$ to $A_i'$ linearly rescales utility: $u'(x) = a \cdot u(x) + b$.
- *Spectral risk measures:* each individual has a probability $p$ s.t. higher risk is not tolerated.
- *Conclusion:* we gauge an action by the value $F^{-1}(p)$ for which probability of smaller benefits is $\le p$.
- *The total benefit* to a group is $\int \varphi(p) \cdot F^{-1}(p)\, dp$, where $\varphi(p)$ is proportion of people with risk tolerance $p$.

# 5. Practical Situation: Sample

- Both the utility and the spectral risk measure approaches allow arbitrary probability distributions.
- In practice, we usually do not know corresponding probability distribution.
- We usually only have a *sample* $x_1, \ldots, x_n$ of the corresponding monetary amounts.
- It is natural to build a histogram based on these values.
- Equivalently, we build an "empirical" distribution in which we have each $x_i$ has equal probability $1/n$.
- It is well known that when the sample size increases, this empirical distribution converges to the actual one.
- How will both approaches look like for this empirical distribution?

# 6. Utility Approach on the Example of a Sample

- *First idea:* expected *utility* value $\overline{u} = \dfrac{1}{n} \cdot \sum_{i=1}^{n} u(x_i)$.
- *Problem:* the risk measures approach provides an equivalent *monetary* value.
- *Solution:* reformulate the utility approach so that it leads to a monetary value.
- *Economic interpretation:* the value of an action $a$ is the amount $x$ that a person pays to participate in $a$.
- *In detail:* the value $x$ for which the expected utility is

$$\frac{1}{n} \cdot u(x_1 - x) + \ldots + \frac{1}{n} \cdot u(x_n - x) = u(0).$$

- *Comment:* we can re-scale utility to $u(0) = 0$, so

$$u(x_1 - x) + \ldots + u(x_n - x) = 0.$$

# 7. Spectral Risk Measure on the Example of a Sample

- *Reminder:* $x_{\rm sp} = \int \varphi(p)\cdot F^{-1}(p)\,dp.$
- *For a sample distribution:*

$$x_{\rm sp} = \frac{1}{n}\cdot\sum_{i=1}^{n}\varphi\left(\frac{i}{n}\right)\cdot x_{(i)}, \tag{1}$$

where $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$ is a sorted sample.

- *Reminder:* for utility, we get

$$u(x_1 - x_{\rm ut}) + \ldots + u(x_n - x_{\rm ut}) = 0. \tag{2}$$

- *Problem:*
  - given $u(x)$, find $\varphi(p)$ for which $x_{\rm sp} \approx x_{\rm ut}$;
  - given $\varphi(p)$, find $u(x)$ for which $x_{\rm sp} \approx x_{\rm ut}$.

## 8. A Similar Problem Is Already Solved In Robust Statistics

- *Robust statistics:* making estimates under partial information about the probability distribution $f(x)$.
- *Typical techniques:* use statistical techniques corresponding to some pdf $f_0(x)$.
- *M-methods:* Max Likelihood $\max\limits_a L \stackrel{\text{def}}{=} \prod\limits_{i=1}^n f_0(x_i - a)$.
- *Specifics:* $\max\limits_a \ln(L) = \sum\limits_{i=1}^n \ln(f_0(x_i - a)) \Rightarrow \dfrac{dL}{da} = 0$, so $\sum\limits_{i=1}^n U(x_i - a) = 0$, where $U(x) = -(\ln(f_0))' = -\dfrac{f_0'(x)}{f_0(x)}$.
- *L-estimate:* $a_L = \dfrac{1}{n} \cdot \sum\limits_{i=1}^n m\left(\dfrac{i}{n}\right) \cdot x_{(i)}$ for some $m(p)$.
- *Observation:* these are exactly our formulas for utility and spectral risk measures.

# 9. Our Idea

- *Reminder:* we have estimates:
  - $\sum\limits_{i=1}^{n} U(x_i - a_M) = 0$, where $U(x) = -\dfrac{f_0'(x)}{f_0(x)}$;
  - $a_L = \dfrac{1}{n} \cdot \sum\limits_{i=1}^{n} m\left(\dfrac{i}{n}\right) \cdot x_{(i)}$.
- *Fact:* in robust statistics, it is known how, given $f_0(x)$, to find $m(p)$ for which $a_M$ and $a_L$ are asympt. close:
  1. we compute the cumulative distribution function $F_0(x)$ as $F_0(x) = \int_{-\infty}^{x} f_0(t)\,dt$;
  2. we find the auxiliary function $M(p)$ as $M(F_0(x)) = -(\ln(f_0(x))''$;
  3. we normalize $m(p) = \dfrac{M(p)}{\int_0^1 M(q)\,dq}$.
- *Our idea:* use this relation in economic applications.

# 10. Correspondence Between M- and L-Estimates

- *Situation:* we do not know the exact shape of a probability density function $f(x)$.
- *What we know:* the *class* $F_0$ of possible shapes.
- *In detail:* we know that $f(x) = f_0(x-a)$ for some $f_0(x) \in F_0$ and $a \in R$.
- *Criteria:* minimize *worst-case* accuracy

$$q_M(U) = \sup_{f_0 \in F} E_{f_0}[(a_M(U)-a)^2]; \;\; q_L(m) = \sup_{f_0 \in F} E_{f_0}[(a_L(m)-a)^2].$$

- *Solution:*
  - find the probability distribution $f_0(x) \in F_0$ with the smallest Fisher information

$$I(f_0) \stackrel{\text{def}}{=} \int \left(\frac{f_0'(x)}{f_0(x)}\right)^2 \cdot f_0(x)\, dx;$$

  - use M- and L-estimates optimal for this $f_0(x)$.

# 11. $u(x) \Leftrightarrow \varphi(p)$: Resulting Algorithms

- Given $u(x)$, we can find $\varphi(p)$ as follows:
  - compute $f_0(x) = \exp\left(-\int_c^x u(t)\,dt\right)$;
  - *comment:* since $u(x) = (\ln(f_0))'$, we have $\ln f_0(x) = \int_c^x U(t)\,dt$ and $f_0(x) = \exp(\ln(f_0(x)))$;
  - compute $F_0(x) = \int_{-\infty}^x f_0(t)\,dt$;
  - find $M(p) = u'(F_0^{-1}(p))$;
  - compute $I \stackrel{\text{def}}{=} \int_0^1 M(q)\,dq$ and $\varphi(p) = \dfrac{M(p)}{I}$.
- Given $\varphi(p)$, we can find $u(x)$ as follows:
  - first, we find $F_0(x)$ and $I$ from the equation
$$I \cdot \varphi(F_0(x)) = -(\ln(F_0'(x)))'';$$
  - then, we find $f_0(x) = F_0'(x)$ and $u(x) = -\dfrac{f_0'(x)}{f_0(x)}$.

## 12. First Example

- *Case study:* utility is proportional to the monetary value $U(x) = x$ (risk-neutral).
- Here, $\int_c^x U(t)\,dt = \dfrac{1}{2} \cdot x^2$.
- $f_0(x) = \exp\left(-\int_c^x U(t)\,dt\right) = \exp\left(-\dfrac{1}{2} \cdot x^2\right) \sim N(0, 1)$.
- Hence, $F_0(x) = \int_{-\infty}^x f_0(t)\,dt$ is the cumulative distribution function of a normal distribution.
- Here, $U'(x) = x$, so $M(p) = U'(F_0^{-1}(p)) = 1$.
- The integral $I$ of $M(p) = 1$ over the interval $[0, 1]$ is 1.
- Therefore, $\varphi(p) = \dfrac{M(p)}{I} = 1$.
- *Conclusion:* all possible values $p$ are equally probable.

# 13. Second Example

- *Case study:* utility is bounded by some value $c$, $U(x) = \max[-c_0, \min(c_0, x)]$:
  - $U(x) = -c_0$ for all $x \le -c_0$,
  - $U(x) = x$ for all $x \in [-c_0, c_0]$, and
  - $U(x) = c_0$ for all $x \ge c_0$,
- *Meaning:* very strong gains and very severe losses are ignored by the decision maker.
- *Solution:* for some $\alpha_0$, we have $\varphi(p) = \dfrac{1}{1 - 2\alpha_0}$ for all $p \in [\alpha_0, 1 - \alpha_0]$.
- *Meaning:* very small ($p < \alpha_0$) and very high ($p > 1 - \alpha_0$) values of $p$ can also be ignored.
- *Explanation:* such probabilities $p$ only affect a decision when combined with very large gains or losses.

# 14. Advantages of the New Correspondence Formulas

- *Case study:* exponential $u(x) = 1 - e^{-k\cdot x}$ and power $u(x) = x^{1-\gamma}$ utility functions.
- *Heuristic approach:* exponential and power $\varphi(p)$.
- *Problem:* e.g., power $u(x)$ and power $\varphi(p)$ lead to different behaviors.
- *New approach:* they are no longer linked.
- *Details:* for $u(x) = x^{1-\gamma}$, we get

$$f_0(x) = \exp\left(-\int_c^x t^{1-\gamma}\,dt\right) = A\cdot\exp\left(-\text{const}\cdot x^{2-\gamma}\right).$$

- So, $F_0(x) = \int^x f_0(x) = A\cdot\int^x \exp\left(-\text{const}\cdot x^{2-\gamma}\right)$, and

$$M\left(A\cdot\int^x \exp\left(-\text{const}\cdot x^{2-\gamma}\right)\right) = (1-\gamma)\cdot x^{-\gamma}.$$

- *Easy to check:* a power f-n $M(p)$ is *not* a solution.

## 15. Conclusions

- Traditional decision theory describes human behavior and human preferences in terms of utility functions.
- In many economic situations, spectral risk measures approach leads to a reasonable description.
- In each of these approaches, we first need to empirically find the corresponding function:
  - utility function $u(x)$ in the traditional approach;
  - the weighting function $\varphi(p)$ for spectral risk measures.
- Both approaches provide a reasonable description of the same actual economic behavior.
- It is thus desirable to be able, given $u(x)$, to find an appropriate $\varphi(p)$ (and vice versa).

## 16. Conclusions (cont-d)

- *Reminder:* it is desirable to be able, given $u(x)$, to find an appropriate $\varphi(p)$ (and vice versa).
- Some empirical rules for such transition have been proposed.
- These rules are purely heuristic and sometimes inaccurate.
- In the present paper,
  - we recall how both the utility and the risk measure approaches can be reformulated in statistical terms;
  - we use these reformulations to provide a statistically justified transition between $u(x)$ and $\varphi(p)$;
  - thus, we avoid pairing functions $u(x)$ and $\varphi(p)$ that lead to different behaviors.

# 17. Acknowledgments

This work was supported in part by:

- by National Science Foundation grants HRD-0734825 and DUE-0926721,
- by Grant 1 T36 GM078000-01 from the National Institutes of Health, and
- by Grant 5015 from the Science and Technology Centre in Ukraine (STCU), funded by European Union.