How to Bargain: An Interval Approach

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1. Need to Bargain: Typical Situations

- If you want to *buy* a house with a list price $\overline{a}$, what offer a should you make?
- If you are already in a negotiating process,
  - your previous offer was $\underline{a}$,
  - the seller's last offer was $\overline{a} > \underline{a}$,
  - what next offer a should you make?
- If you want to *sell* a house, and a potential buyer made an offer $\underline{a}$, what counter-offer a should you make?
- If you are already in a negotiating process,
  - your previous offer was $\overline{a}$,
  - the buyer's related counter-offer was $\underline{a} < \overline{a}$,
  - what next offer a should you make?

2. Need to Bargain: Other Examples

Other examples of bargaining:

- negotiating a *bankruptcy* deal:
  - if the original debt was $\overline{a}$, and
  - the company going bankrupt is offering to pay an amount $\underline{a} < \overline{a}$,
  - what is a reasonable next offer?
- negotiating for a *salary* with a new hire;
- negotiating between an employer and an *insurance* company for the best way to provide insurance to the employees;
- in an *auction*, when the previous bid was $\underline{a}$, what next bid should you make?

3. Commonsense Solutions

- A usual advise for the first offer is to offer a portion $a = k \cdot \overline{a}$ of the asking price, for some $k \in (0, 1)$.
- The exact value of the coefficient k depends on the situation:
  - when buying a house in the US, 70-80% is usually appropriate;
  - when buying a souvenir in an oriental bazaar, 1/3 (or even 1/4) may be appropriate.
- A usual advise to use in the middle of negotiations is, e.g., to split the difference, i.e., to select $a = (\underline{a} + \overline{a})/2$.
- In most cases, the recommended offer a is a linear function of the bounds $\underline{a}$ and $\overline{a}$.

4. Usual Game Theory Approach to Bargaining: Reminder

- In general, economic situations like this, with conflict of interest, are handled by game theory.
- Traditional game-theoretic solution concepts:
  - select the *set* of reasonable outcome, and
  - leave the exact choice to the participants' bargaining skills.
- Nash's bargaining solution:
  - the first game-theoretic bargaining solution was proposed by the Nobelist John Nash;
  - he proposed to select an alternative with the largest product of utility gains;
  - under reasonable assumptions, this leads to a linear function of the bounds $\underline{a}$ and $\overline{a}$.

5. From Nash's Bargaining Solution to Actual Bargaining: Successes and Limitations

- *Problem:* Nash's bargaining solution does not explain how exactly we should bargain.
- *Solution:* a more sophisticated game-theoretic analysis leads to bargaining recommendations $a = f(\underline{a}, \overline{a})$.
- *Details:* the resulting function f is linear.
- *Reminder:* there are many solution concepts and utility functions.
- *Problem:* under some other concepts and/or assumptions, we get a non-linear recommendation $a = f(\underline{a}, \overline{a})$.
- *Objective:* recommendations which do not depend:
  - on the specific (and somewhat arbitrary) choice of a solution concept and/or
  - on difficult-to-verify assumptions about utility f-s.

6. First Offer Problem: Analysis

- *Reminder:* produce an offer $a' = f(a)$ based on the asking price $a \geq 0$.
- *Situation:* we want to buy (or sell) two houses at the same time, with asking prices a and b.
- If we treat these houses as *separate purchases*, we should offer $f(a)$ for the 1st house and $f(b)$ for the 2nd house.
- Thus, the total amount of the offer is $f(a) + f(b)$.
- If we view the two houses as a *single purchase*, with the initial price $t = a + b$, we offer $f(t) = f(a + b)$.
- It makes sense to require that the total amount offered should not depend on how we treat the situation:

$$f(a + b) = f(a) + f(b).$$

7. First Offer Problem: Result

Definition. *By a* unary recommendation function, *we mean a function $f : \mathbb{R} \to \mathbb{R}$ that maps every non-negative number a into a non-negative value $f(a)$ for which*

$$f(a+b) = f(a) + f(b).$$

Proposition. *Every unary recommendation function has the form $f(a) = k \cdot a$ for some real number $k \geq 0$.*

Comment. The choice of k depends on whether we consider a buyer or a seller problem:

- for a buyer, $f(\overline{a}) \leq \overline{a}$, so $k \leq 1$;
- for a seller, $f(\underline{a}) \geq \underline{a}$, so $k \geq 1$.

8. Selecting an Offer in the Middle of a Bargaining Process: Analysis

- *Reminder:* produce an offer $a = f(\underline{a}, \overline{a})$ based on the current offers $\underline{a} < \overline{a}$.
- *Situation:* we want to buy (or sell) two houses at the same time, with offers $\underline{a} < \overline{a}$ and $\underline{b} < \overline{b}$.
- If we treat these houses as *separate purchases*, we should offer $f(\underline{a}, \overline{a})$ for the 1st house and $f(\underline{b}, \overline{b})$ for the 2nd.
- Thus, the total amount of the offer is $f(\underline{a}, \overline{a}) + f(\underline{b}, \overline{b})$.
- If we view the houses as a *single purchase*, with the current offers $\underline{t} \stackrel{\text{def}}{=} \underline{a} + \underline{b} < \overline{t} \stackrel{\text{def}}{=} \overline{a} + \overline{b}$, we offer $f(\underline{t}, \overline{t})$.
- It makes sense to require that the total amount offered should not depend on how we treat the situation:

$$f(\underline{a} + \underline{b}, \overline{a} + \overline{b}) = f(\underline{a}, \overline{a}) + f(\underline{b}, \overline{b}).$$

9. Selecting an Offer in the Middle of a Bargaining Process: Result

Definition. *By a* binary recommendation function, *we mean a function* $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *that maps every pair* $(\underline{a},\overline{a})$ *of non-negative numbers with* $\underline{a}\le\overline{a}$ *into a non-negative value* $f(\underline{a},\overline{a})\in[\underline{a},\overline{a}]$ *and for which*

$$f(\underline{a}+\underline{b},\overline{a}+\overline{b})=f(\underline{a},\overline{a})+f(\underline{b},\overline{b}).$$

Proposition. *Every binary recommendation function has the form*

$$f(\underline{a},\overline{a})=k\cdot\overline{a}+(1-k)\cdot\underline{a}$$

for some real number $k\in[0,1]$.

10. Relation to Intervals and Interval Computations

- The pair $(\underline{a}, \overline{a})$ of non-negative numbers with $\underline{a} \le \overline{a}$ represents an *interval* $\mathbf{a} = [\underline{a}, \overline{a}]$.
- For intervals, addition is defined as

$$\mathbf{a} + \mathbf{b} \stackrel{\text{def}}{=} \{a + b : a \in \mathbf{a}, b \in \mathbf{b}\}.$$

- *Known result:* this leads to

$$[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}].$$

- Thus, in interval terms, the additivity requirement

$$f(\underline{a} + \underline{b}, \overline{a} + \overline{b}) = f(\underline{a}, \overline{a}) + f(\underline{b}, \overline{b})$$

takes the form

$$f(\mathbf{a} + \mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b}).$$

11. Relation to Decision Making Under Uncertainty

- Suppose that we only know that the price of an object A is between $\underline{a}$ and $\overline{a}$.
- What is a reasonable price $a = f(\underline{a}, \overline{a})$ to pay for A?
- If we buy two objects, the fair price should not depend on whether we consider them separately or together:

$$f(\mathbf{a} + \mathbf{b}) = f(\mathbf{a}) + f(\mathbf{b}).$$

- Thus, according to our proposition, the fair price is $f(\underline{a}, \overline{a}) = k \cdot \overline{a} + (1 - k) \cdot \underline{a}$.
- This is the optimism-pessimism criterion proposed by another Nobelist L. Hurwicz:
  - $k = 1$ and $a = \overline{a}$ is the most optimistic case;
  - $k = 0$ and $a = \underline{a}$ is the most pessimistic case.

12. Should Recommendation Depend on Pre-History?

Definition. *Let T be a positive integer. By a* T-ary recommendation function, *we mean a function f that maps every tuple $(\mathbf{a}_0, \ldots, \mathbf{a}_T)$ of $T+1$ non-negative intervals satisfying the condition $\mathbf{a}_{t+1} \subseteq \mathbf{a}_t$ into a non-negative value*

$$f(\mathbf{a}_0, \ldots, \mathbf{a}_T) \in \mathbf{a}_T \text{ for which}$$

$$f(\mathbf{a}_0 + \mathbf{b}_0, \ldots, \mathbf{a}_T + \mathbf{b}_T) = f(\mathbf{a}_0, \ldots, \mathbf{a}_T) + f(\mathbf{b}_0, \ldots, \mathbf{b}_T).$$

Proposition. *For every T, every T-ary recommendation function depends only on $\mathbf{a}_T$: $f(\mathbf{a}_0, \ldots, \mathbf{a}_T) = f(\mathbf{a}_T)$.*

Comment. The resulting recommendation depends, Markov-process-style, only on the latest offer $\mathbf{a}_T$.

13. Proof: Main Idea

- The above result seems counter-intuitive, so we show the proof to make it more convincing.
- Due to additivity, the function $f(\mathbf{a}_0, \ldots, \mathbf{a}_T)$ linearly depends on all its variables.
- Thus,

$$f(\mathbf{a}_0, \ldots, \mathbf{a}_T) = \sum_{t=0}^{T} \ell_t \cdot \underline{a}_t + \sum_{t=0}^{T} u_t \cdot \overline{a}_t$$

  for some coefficients ℓ_t and u_t.
- We require that $f(\mathbf{a}_0, \ldots, \mathbf{a}_T) \in \mathbf{a}_T$ for all $\mathbf{a}_0$,..., $\mathbf{a}_T$.
- For the case when $\mathbf{a}_T$ is a degenerate interval $\mathbf{a}_T = [a_T, a_T]$, this means that $f(\mathbf{a}_0, \ldots, \mathbf{a}_T) = a_T$.
- Thus, the coefficients ℓ_t and u_t corresponding to $t < T$ should be equal to 0. Q.E.D.

14. Previous Derivation of Hurwicz F-la: Reminder

- We derived the Hurwicz formula from the requirement the $f(\underline{a}, \overline{a})$ be shift- and scale-invariant.
- Shift-invariance means that:
  - if we add a thing of a fixed cost c to the object,
  - the recommended price should increase by this price c:

$$f(\underline{a} + c, \overline{a} + c) = f(\underline{a}, \overline{a}) + c.$$

- Shift-invariance is a particular case of additivity.
- Scale-invariance means that:
  - if we use different (λ times smaller) monetary units,
  - recommendations should stay the same:

$$f(\lambda \cdot \underline{a}, \lambda \cdot \overline{a}) = \lambda \cdot f(\underline{a}, \overline{a}).$$

- Similarly, shift- and scale-invariance imply the bargain formula $f(\underline{a}, \overline{a}) = k \cdot \overline{a} + (1 - k) \cdot \underline{a}$.

15. Previous Derivation Cannot Be Extended to the Analysis of Dependence on Pre-History

- We derived the formulas for the bargaining from additivity;
- For $T = 0$ (no dependence on pre-history), the same formula can be derived from shift- and scale-invariance.
- However, already for $T = 1$, shift- and scale-invariance is not sufficient.
- *Counter-example:*
  - the recommendation

$$f([\underline{a}_0, \overline{a}_0], [\underline{a}_1, \overline{a}_1]) = \max\left(\underline{a}_1, \min\left(\overline{a}_1, \frac{\underline{a}_0 + \overline{a}_0 + \underline{a}_1 + \overline{a}_1}{4}\right)\right)$$

    is shift- and scale-invariant;
  - however, this recommendation depends on $\mathbf{a}_0$ as well: e.g.,

$$f([0, 3], [1, 2]) = 1.5 \neq f([0, 4], [1, 2]) = 1.75.$$

16. Application to Auctions

We can consider the dependence of the next bid on the previous bids $a_0 \le a_1 \le a_2 \le \ldots \le a_T$. The result is different.

Definition. *Let T be a positive integer. By a T-ary* auction recommendation function, *we mean a function f that maps every tuple $(a_0, \ldots, a_T)$ of $T+1$ non-negative numbers satisfying the condition $a_t \le a_{t+1}$ into a non-negative value $f(a_0, \ldots, a_T) \ge a_T$ for which*

$$f(a_0 + b_0, \ldots, a_T + b_T) = f(a_0, \ldots, a_T) + f(b_0, \ldots, b_T).$$

Proposition. *For every T, every T-ary recommendation function has the form*

$$f(a_0, \ldots, a_T) = c_0 \cdot a_0 + \sum_{t=1}^{T} c_t \cdot (a_t - a_{t-1})$$

for some values $c_0 \ge 1$, $c_1 \ge 1$, $\ldots$, $c_T \ge 1$.

17. How to Find the Equilibrium

- As a result of the negotiations process, we get converging offers and counter-offers.
- In the limit, we reach an "equilibrium" – the price to which both the buyer and the seller agree.
- We consider the cases that start with the seller's offer, so $[\underline{a}(0), \overline{a}(0)] = [0, \overline{a}_0]$.
- On each iteration, the buyer offers
$$\underline{a}_t = k_b \cdot \overline{a}_{t-1} + (1 - k_b) \cdot \underline{a}_{t-1}.$$
- Now, the seller has an interval $[\underline{a}_t, \overline{a}_{t-1}]$, and selects
$$\overline{a}_t = k_s \cdot \overline{a}_{t-1} + (1 - k_s) \cdot \underline{a}_t.$$
- One can prove that the process converges, and the resulting equilibrium is $a = \dfrac{1-k}{1-k+k_b} \cdot \overline{a}_0$, where
$$k \stackrel{\text{def}}{=} k_s + k_b - k_s \cdot k_b.$$

18. What is the Purpose of Negotiations in the First Place?

- *Question:* the negotiation ends up with a single value a, so why not come up with this value right away?
- *Answer:* some buyers may value their time more and stop negotiations earlier.
- *Analysis* shows that at t-th iteration, the seller's price is $\overline{a}_t = a + \Delta \cdot k_0^t$, where $\Delta \stackrel{\text{def}}{=} \overline{a}_0 - a$ and $k_0 \stackrel{\text{def}}{=} k - k_b$.
- *The buyer's loss* at moment t is $\Delta \cdot k_0^t + w_b \cdot t$, where w_b is the cost of his/her time.
- *Recommendation:* the buyer stops when the loss is minimal:
$$t = \frac{\ln(w_b) - \ln(\Delta) - \ln(|\ln(k_0)|)}{\ln(k_0)}.$$
- *Conclusion:* the more the buyer values his/her time, the earlier the buyer stops bargaining.

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