Why Clayton & Gumbel Copulas: A Symmetry-Based Explanation

Vladik Kreinovich[1], Hung T. Nguyen[2,3], and Songsak Sriboonchitta[3]

[1]University of Texas at El Paso, USA
[2]Department of Mathematics, New Mexico State University, USA
[3]Department of Economics, Chiang Mai University, Thailand
contact email vladik@utep.edu

1. Copulas Are Needed

- Traditionally, in statistics the dependence between random variables η, ν, ..., is described by *correlation*.
- A *Gaussian* joint distribution can be uniquely reconstructed from correlation and marginals

$$F_\eta(x) \stackrel{\text{def}}{=} P(\eta \le x), \quad F_\nu(y) \stackrel{\text{def}}{=} P(\nu \le y).$$

- In many practical situations, e.g., in economics, the distributions are often *non-Gaussian*.
- For non-Gaussian variables (η, ν), in general, marginals and correlation are *not sufficient*.
- To reconstruct a joint distribution, we need to know a *copula*, i.e., by a function $C(u, v)$ for which

$$P(\eta \le x \,\&\, \nu \le y) = C(F_\eta(x), F_\nu(y)).$$

- Usually, *Archimedean* copulas are used, i.e., f-s $C(u, v) = \psi(\psi^{-1}(u) + \psi^{-1}(v))$ for some $\psi : [0, \infty) \to (0, 1]$, $\psi \downarrow$.

2. Most Efficient Archimedean Copulas

- In econometric applications, the following three classes of Archimedean copulas turned out to be most efficient:
  - the Frank copulas

$$C(u,v)=-\frac{1}{\theta}\cdot\ln\left(1-\frac{(1-\exp(-\theta\cdot u))\cdot(1-\exp(-\theta\cdot v))}{1-\exp(-\theta)}\right);$$

  - the Clayton copulas

$$C(u,v)=\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta};$$

  - the Gumbel copulas

$$C(u,v)=\exp\left(\left((-\ln(u))^{-\theta}+(-\ln(v))^{-\theta}-1\right)^{-1/\theta}\right).$$

- Frank copulas are useful as the only Archimedean copulas which satisfy the natural condition

$$C(u,v)+C(u,1-v)+C(1-u,v)+C(1-u,1-v)=1:$$

$$P(U\,\&\,V)+P(U\,\&\,\neg V)+P(\neg U\,\&\,V)+P(\neg U\,\&\,\neg V)=1.$$

3. Why Clayton and Gumbel Copulas?

- *Known fact:* there are *many* different classes of copulas.
- *Main question:* why did the above two classes turned out to be the most efficient in econometrics?
- *Auxiliary question:*
  - What if these copulas are not sufficient?
  - Which classes should we use?
- *What we do:* we show that natural *symmetry*-based ideas answer both questions:
  - symmetry ideas explain the efficiency of the above classes of copulas, and
  - symmetry ideas can lead us, if necessary, to more general classes of copulas.

4. Why Symmetries

- Symmetries are ubiquitous in modern physics, because most of our knowledge is based on symmetries.
- *Example:* we drop a rock, and it always falls down with an acceleration of 9.81 m/sec^2.
- *Meaning:* no matter how we shift or rotate, the fundamental laws of physics do not change.
- This idea is actively used in modern physics.
- Starting with quarks, physical theories are often formulated in terms of symmetries (not diff. equations).
- Fundamental physical equations – Maxwell's, Schrödinder's, Einstein's – can be uniquely derived from symmetries.
- In view of efficiency of symmetries in physics, it is reasonable to use them in other disciplines as well.

5. Basic Symmetries

- *Basic* symmetries that come from the fact that the numerical value of a physical quantity depends:
  - on the choice of the measuring unit (*scaling*) and
  - on the choice of a starting point (*shift*).
- *Scaling:* when we replace a unit by the one which is λ times smaller, we get new values $x' = \lambda \cdot x$.
- *Example:* 2 m is the same as $2 \cdot 100 = 200$ cm.
- *Shift:* when we change a starting point to a one which is s units earlier, we get new values $x' = x + s$.
- *Example:* 25 C is the same as $25 + 273.16 = 298.16$ K.
- *Observation:* many physical formulas are *invariant* relative to these symmetries.

6. Example of Invariance

- *Example:* power law $y = x^a$ has the following invariance property:
  - if we change a unit in which we measure x,
  - then get the exact same formula – provided that we also appropriately changing a measuring unit for y:

$$f(\lambda \cdot x) = \mu(\lambda) \cdot f(x).$$

- *Interesting fact:* power law is the only scale-invariant dependence:
  - differentiating both sides by λ, we get

$$x \cdot f'(\lambda \cdot x) = \mu'(\lambda) \cdot f(x);$$

  - for $\lambda = 1$, we get $x \cdot \dfrac{df}{dx} = \mu_0 \cdot f$, hence $\dfrac{df}{f} = \mu_0 \cdot \dfrac{dx}{x}$;
  - integrating, we get $\ln(f) = \mu_0 \cdot \ln(x) + C$, i.e., power law $f(x) = e^C \cdot x^{\mu_0}$.

7. Invariant Functions Corr. to Basic Symmetries

- A diff. f-n $f(x)$ is called *scale-to-scale invariant* if for every λ, there exists a μ for which $f(\lambda \cdot x) = \mu \cdot f(x)$.
- **Thm.** A function $f(x)$ is scale-to-scale invariant $\Leftrightarrow$ it has the form $f(x) = A \cdot x^a$ for some A and a.
- A diff. f-n $f(x)$ is called *scale-to-shift invariant* if for every λ, there exists an s for which $f(\lambda \cdot x) = f(x) + s$.
- **Thm.** A function $f(x)$ is scale-to-shift invariant $\Leftrightarrow$ it has the form $f(x) = A \cdot \ln(x) + b$ for some A and b.
- **Thm.** A function $f(x)$ is shift-to-scale invariant $\Leftrightarrow$ it has the form $f(x) = A \cdot \exp(k \cdot x)$ for some A and k.
- **Thm.** A function $f(x)$ is shift-to-shift invariant $\Leftrightarrow$ it has the form $f(x) = A \cdot x + c$ for some A and c.
- A function is called *invariant* $\Leftrightarrow$ it is scale-to-scale, scale-to-shift, shift-to-scale, or shift-to-shift invariant.

8. Not All Physical Dependencies Are Invariant

- Physical dependencies of z on x are often indirect:
  - the quantity x influences some intermediate quantity y, and
  - the quantity y influences z.
- The resulting mapping $x \to z$ is a composition $z = g(f(x))$ of mappings $y = f(x)$ and $z = g(y)$.
- The basic mappings $f(x)$ and $g(y)$ are often invariant.
- However, the composition of inv. f-ns may not be inv.: e.g., $f(x) = x^a$, $g(y) = \exp(y)$, $z = \exp(x^a)$ is not inv.
- It is thus reasonable to look for functions which are compositions of two invariant functions.
- If necessary, we can look for compositions of three, etc.
- Let us apply this approach to our problem of finding appropriate copulas.

9. Why Scalings Can Be Applied to Probabilities

- At first glance, for probabilities, 0 is a natural starting point, and 1 is a natural measuring unit.
- However, we often have *conditional* probabilities.
- Example: a stock is called stable if its price drastically changes only when the market changes.
- If in N days, the stick has drastic change on c days, we can gauge its stability as $p = \dfrac{c}{N}$.
- Alternatively, we can use $p' = \dfrac{c}{n}$, where $n \ll N$ is the number of days when the market drastically changed.
- The two resulting probabilities differ by a multiplicative constant $p' = \lambda \cdot p$, where $\lambda \stackrel{\text{def}}{=} \dfrac{n}{N}$.
- So, for econometric probabilities, scaling makes sense.

10. Why Shifts Can Be Applied to Probabilities

- Suppose that we have a stock which:
  - always fluctuates when the market changes *and*
  - also sometimes experiences drastic changes of its own.
- How can we estimate the stability of this stock?
- One way is use the same ratio $p = \dfrac{c}{N}$ as before.
- It also makes sense to only consider days when the market itself was stable, i.e., take $p' = \dfrac{c - n}{N - n}$.
- Since $n \ll N$, we have $p' \approx \dfrac{c - n}{N} = p + s$, where $s \stackrel{\text{def}}{=} -\dfrac{n}{N}$.
- Thus, in econometric applications, shifts also make sense for probabilities.

11. Answer to the Main Question: Why Clayton and Gumbel Copulas

- **Thm.** The only Archimedean copula with an inv. generator is the copula corr. to independence:

$$C(u,v) = u \cdot v.$$

- **Thm.** The only Archimedean copulas in which a generator is a *composition* of two invariant functions are:
  - the Clayton copulas
  $$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta};$$
  - the Gumbel copulas
  $$C(u,v) = \exp\left(\left((-\ln(u))^{-\theta} + (-\ln(v))^{-\theta} - 1\right)^{-1/\theta}\right);$$
  - the copulas
  $$C(u,v) = \frac{1}{L}\exp\left(\frac{1}{\ell} \cdot \ln(u \cdot L) \cdot \ln(v \cdot L)\right), \text{ with } \ell = \ln(L).$$

12. Idea of the Proofs

- There are four types of invariant functions:
  - scale-to-scale invariant $f(x) = A \cdot x^a$;
  - scale-to-shift invariant $f(x) = A \cdot \ln(x) + b$;
  - shift-to-scale invariant $f(x) = A \cdot \exp(k \cdot x)$;
  - shift-to-shift invariant $f(x) = A \cdot x + c$.
- Generator f-ns should have $\psi(0) = 1$ and $\psi(\infty) = 0$.
- So, the only invariant generator function is $\psi(x) = A \cdot \exp(k \cdot x)$; for this f-n, $\psi^{-1}(u) = \frac{1}{k} \cdot \ln(u)$, thus:

$$C(u, v) = \psi(\psi^{-1}(u) + \psi^{-1}(v)) =$$

$$\exp\left(k \cdot \left(\frac{1}{k} \cdot \ln(u) + \frac{1}{k} \cdot \ln(v)\right)\right) =$$

$$\exp(\ln(u) + \ln(v)) = \exp(\ln(u \cdot v)) = u \cdot v.$$

13. Idea of the Proofs (cont-d)

- For composition $z = g(f(x))$, we have:
  - four types of invariant functions $f(x)$ and
  - four types of invariant functions $g(y)$.
- So we have $4 \times 4 = 16$ possible compositions

$$\psi(x) = g(f(x)).$$

- We consider these 16 combinations one by one.
- Out of 16 combinations, the only new generators are:
  - $\psi(x) = B \cdot \exp(k \cdot A \cdot x^a)$ corr. to Gumbel copula,
  - $\psi(x) = B \cdot (A \cdot x + b)^a$ corr. to Clayton copula,
  - $\psi(x) = \exp(\ell \cdot (\exp(k \cdot x) - 1))$ corresponding to the new copula

$$C(u,v) = \frac{1}{L} \exp\left(\frac{1}{\ell} \cdot \ln(u \cdot L) \cdot \ln(v \cdot L)\right), \text{ with } \ell = \ln(L).$$

14. Answer to the Auxiliary Question

- *Reminder:* we also had an auxiliary question:
  - when the current copulas are not sufficient,
  - which copulas should we then use?
- Our symmetry-based answer to this question:
  - we should first try Archimedean copulas whose generator function is a composition of 3 inv. f-ns,
  - if needed, we should move to Archimedean copulas whose generator f-n is a composition of 4 inv. f-ns, etc.
- *Example:* generator $z = \psi(x)$ of Frank's copula is a composition of 3 invariant functions:

$$z = \psi(x) = -\frac{1}{\theta} \cdot \ln(1 - (1 - \exp(-\theta)) \cdot \exp(-x));$$

$$y = f(x) = (1 - e^{-\theta}) \cdot e^{-x},\ z = g(y) = 1 - y,\ t = h(z) = -\frac{1}{\theta} \cdot \ln(z).$$

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