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# From Mean and Median Income to the Most Adequate Way of Taking Inequality Into Account

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# 1. Outline

- How can we compare the incomes of two different countries or regions?
- At first glance, it is sufficient to compare the mean incomes.
- However, this is known to be not a very adequate comparison.
- A more adequate description of economy is the *median* income.
- However, the median is also not always fully adequate.
- We use Nash's bargaining solution to come up with the most adequate measure of "average" income.
- On several examples, we illustrate how this measure works.


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## 2. Mean Income and Its Limitations

- How can we compare the economies of two countries (or two regions) A and B?
- At first glance, we can divide the total income by the number of people, and get mean incomes $\mu_A$ and $\mu_B$.
- If $\mu_A > \mu_B$, we conclude that A's economy is better.
- Problem: What if Bill Gates walks into a bar?
- On average, everyone becomes a millionaire.
- If a billionaire moves into a poor country, its mean income increases but the country remains poor.
- So, when comparing different economies, we also need to take into account income inequality.


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## 3. Medium Income: A More Adequate Measure

- The most widely used alternative to mean is the *median* income $m_A$, the level for which:
  - the income of exactly half of the population is above $m_A$, and
  - the income of the remaining half is below $m_A$.
- This is how the Organization for Economic Cooperation and Development (OECD) compares economies.
- Median resolves some of the mean's problems.
- When Bill Gates walks into a bar, median does not change much.
- In statistical terms:
  - the main problem with the mean is that it is not robust,
  - on the other hand, median *is* robust.


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# 4. Limitations of the Median

- *Example 1:*
  - if the incomes of all the people in the poorer half increase – but do not exceed the previous median,
  - the median remains the same.
- So, median is not adequate measure for describing how well people are lifted out of poverty.
- *Example 2:*
  - if the income of the poorer half drastically decreases,
  - we should expect the adequate measure of "average" income to decrease,
  - but the median remains unchanged.


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# 5. Gauging "Average" Income Reformulated as a Particular Case of Group Decision Making

- Simplest case: all the people in A have the same income $x$, and all the people in B have the same income $y$.
- If $x > y$, this clearly country A is better.
- If $x < y$, then B's economy is better.
- In practice, people from A have different incomes $x_1, \ldots, x_n$, and people people from have different incomes $y_1, \ldots, y_m$.
- So, we find $x$ s.t. for A, incomes $x_1, \ldots, x_n$ are equivalent (in terms of group decision making) to $x, \ldots, x$.
- Similarly, we find $y$ s.t. incomes $y_1, \ldots, y_n$ are equivalent to incomes $y, \ldots, y$.
- If $x > y$, then A's economy is better, else B's economy is better.


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# 6. From the Idea to an Algorithm

- Nash showed that the best idea is to to select the alternative with the largest product of utilities $\prod_{i=1}^{n} u_i$.
- The utility is usually proportional to a power of the money: $u_i = C_i \cdot x_i^a$ for some $a \approx 0.5$.
- Maximizing $\prod_{i=1}^{n} C_i \cdot \prod_{i=1}^{n} x_i^a$ is equivalent to $\prod_{i=1}^{n} x_i \to \max$.
- Thus, the equivalent value $x$ comes from the formula

$$\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} x = x^n.$$

- The resulting value $x$ is the *geometric mean* $x = \sqrt[n]{x_1 \cdot \ldots \cdot x_n}$ of the income values.


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## 7. Resulting Measure of "Average" Income

- *Task:* compare the economies of regions A and B.
- *Given:* incomes $x_1, \ldots, x_n$ in region A.
- *Given:* incomes $y_1, \ldots, y_m$ in region B.
- *Comparison procedure:*
  - compute the geometric averages $x = \sqrt[n]{x_1 \cdot \ldots \cdot x_n}$ and $y = \sqrt[m]{y_1 \cdot \ldots \cdot y_m}$ of the two regions;
  - if $x > y$, then region A is in better economic shape;
  - if $x < y$, then region B is in better economic shape.
- $\ln(\overline{x}) = \ln(\sqrt[n]{x_1 \cdot \ldots \cdot x_n}) = \dfrac{\ln(x_1) + \ldots + \ln(x_n)}{n}$, so: $\overline{x} = \exp(E[\ln(x)]) = \exp\left(\int \ln(x) \cdot f(x)\,dx\right).$
- So, to compare the economies, we need to compare the mean values $E[\ln(x)]$ of the logarithm of the income $x$.


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## 8. Relation Between the New Measure and the Mean Income: An Observation

- It is well known that the geometric mean is always smaller than or equal to the arithmetic mean.
- Geometric mean is equal to the arithmetic mean if and only if all the numbers are equal.
- Thus, the new measure of "average" income is always smaller than or equal to the mean income.
- The new measure is equal to the mean income
  - if and only if all the individual incomes are the same,
  - i.e., if and only if we have perfect equality.


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# 9. First Example: Case of Low Inequality

- *Case:* most incomes are close to one another.
- Thus, most incomes are close to the mean income $\mu$.
- In statistical terms, low inequality means that the standard deviation $\sigma$ is small.
- According to the Taylor series for the logarithm:

$$\ln(x) = \ln(\mu+(x-\mu)) = \ln(\mu)+\frac{1}{\mu}\cdot(x-\mu)-\frac{1}{2\mu^2}\cdot(x-\mu)^2+\ldots$$

- Thus, ignoring higher order terms,

$$E[\ln(x)] = \ln(\mu) - \frac{1}{2\mu^2}\cdot\sigma^2 + \ldots$$

- For $\overline{x} = \exp(E[\ln(x)])$, we similarly get to $\overline{x} = \mu - \frac{\sigma^2}{2\mu}$.

## 10. Analysis of the Formula $\overline{x} = \mu - \dfrac{\sigma^2}{2\mu}$

- The larger the inequality, the larger the standard deviation $\sigma$, and the less preferable is the economy.
- The new measure takes inequality into account, it avoids the ideological ideas of weighing inequality too much.
- An increase in inequality is OK if mean increases more.
- This example is one of the cases which shows that the new measure is more adequate than, e.g., the median.
- For example, if the incomes are normally distributed, then the median simply coincides with the mean.
- So, contrary to our intuitive expectations, the increase in inequality does not worse the median.
- In contrast, the new measure does go down when inequality increases.

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## 11. Second Example: Case of Heavy-Tailed Distributions

- In the 1960s, Benoit Mandelbrot, the author of fractal theory, empirically studied the price fluctuations.
- He showed that large-scale fluctuations follow the Pareto power-law distribution, with the probability density

$$f(x) = A \cdot x^{-\alpha} \text{ for } x \geq x_0, \text{ for } \alpha \approx 2.7 \text{ and some } x_0.$$

- For this distribution, variance is infinite.
- With similar discovery of heavy-tailed laws in other application areas, this has led to *fractal theory*.
- Since then, Pareto distributions have been found in other financial situations.


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# 12. Case of Pareto Distribution (cont-d)

- *Pareto law:* $f(x) = 0$ for $x \le x_0$ and $f(x) = A \cdot x^{-\alpha}$ for $x \ge x_0$.
- Once we know $x_0$ and $\alpha$, we can determine $A$ from the condition that $\int f(x)\,dx = 1$: $A = (\alpha - 1) \cdot x_0^{\alpha-1}$.
- The mean is equal to $\mu = \int x \cdot f(x)\,dx = \dfrac{\alpha - 1}{\alpha - 2} \cdot x_0$.
- The median income is equal to $m = x_0 \cdot 2^{1/(\alpha-1)}$.
- The new measure of "average" income is equal to

$$\overline{x} = \exp(E[\ln(x)]) = x_0 \cdot \exp\left(\frac{1}{\alpha - 1}\right).$$

- When $\alpha \to \infty$, the distribution is concentrated on a single value $x_0$ – i.e., we have absolute equality.
- In this case, all three characteristics – $\mu$, $m$, and $\overline{x}$ – tends to the same value $x_0$.


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# 13. The New Measure $\overline{x}$ May Explain the Power-Law for Income Distribution

- We have estimated $\overline{x} = \exp(\int \ln(x) \cdot f(x)\, dx)$ when the income distribution follows a power law.
- Interestingly, the power law itself can be derived based on this inequality measure.
- Indeed, suppose that all we know about the income distribution is:
  - the value $\overline{x}$, and
  - the lower bound $\delta > 0$ on possible incomes;
  - this lower bound reflects the fact that a human being needs some minimal income to survive.
- There are many possible probability distributions $f(x)$ which are consistent with this information.

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## 14. The New Measure $\overline{x}$ May Explain the Power-Law for Income Distribution (cont-d)

- In such situation, it is reasonable to select a distribution with the largest entropy

$$S \stackrel{\text{def}}{=} -\int f(x) \cdot \ln(f(x))\,dx.$$

- We maximize $S$ under the constraints

$$\exp\left(\int \ln(x) \cdot f(x)\,dx\right) = \overline{x} \text{ and } \int f(x)\,dx = 1.$$

.

- Lagrange multiplier techniques leads to optimizing

$$-\int f(x)\cdot\ln(f(x))\,dx + \lambda_1 \cdot \left(\exp\left(\int \ln(x) \cdot f(x)\,dx\right) - \overline{x}\right) +$$

$$\lambda_2 \cdot \left(\int f(x)\,dx - 1\right).$$


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## 15. The New Measure $\overline{x}$ May Explain the Power-Law for Income Distribution (cont-d)

$$-\int f(x)\cdot\ln(f(x))\,dx+\lambda_1\cdot\left(\exp\left(\int \ln(x)\cdot f(x)\,dx\right)-\overline{x}\right)+$$

$$\lambda_2\cdot\left(\int f(x)\,dx-1\right).$$

- Differentiating this expression with respect to $f(x)$ and equating the derivative to 0, we conclude that

$$-\ln(f(x))-1+\lambda_1\cdot C\cdot\ln(x)+\lambda_2=0.$$

- Here, $C\stackrel{\text{def}}{=}\exp(\int\ln(x)\cdot f(x)\,dx)$ and thus $C=\overline{x}$.
- So, $\ln(f(x))=(\lambda_2-1)+\lambda_1\cdot\overline{x}\cdot\ln(x)$, and $f(x)=A\cdot x^{-\alpha}$, where $A=\exp(\lambda_2-1)$ and $\alpha=-\lambda_1\cdot\overline{x}$.
- Hence, we indeed get the empirically observed power law for income distribution.


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