

What If We Only Have Approximate Stochastic Dominance?

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1. Outline

- In many practical situations,
 - we need to select one of the two alternatives, and
 - we do not know the exact form of the user's utility function – e.g., we only know that it is increasing.
- *Stochastic dominance result*: if for cdfs, $F_1(x) \leq F_2(x)$ for all x , then the 1st alternative is better.
- This criterion works well in many practical situations.
- However, often, $F_1(x) \leq F_2(x)$ for *most* x , but *not* for *all* x .
- In this talk, we show that in such situations:
 - if the set $\{x : F_1(x) > F_2(x)\}$ is sufficiently small,
 - then the 1st alternative is still provably better.

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2. In Finance, We Need to Make Decisions Under Uncertainty

- In financial decision making, we need to select one of the possible decisions.
- For example, we need to decide whether we sell or buy a given financial instrument (share, option, etc.).
- Ideally, we should select a decision which leaves us with the largest monetary value x .
- However, in practice, we cannot predict exactly the monetary consequences of each action.
- Because of the changing external circumstances, the same decision can lead to gains or to losses.
- Thus, we need to make a decision in a situation when we do not know the exact consequences of each action.

3. In Finance, We Usually Have Probabilistic Uncertainty

- Numerous financial transactions are occurring every moment.
- For the past transactions, we know the monetary consequences of different decisions.
- By analyzing past transactions, we can estimate, for each decision, the frequencies of different outcomes x .
- Since the sample size is large, the corresponding frequencies become very close to the actual probabilities.
- Thus, in fact, we can estimate the probabilities of different values x .
- *Comment:* this is not true for new, untested financial instruments, but it is true in most cases.

4. How to Describe the Corresponding Probabilities

- As usual, the corresponding probabilities can be described:
 - either by the probability density function (pdf) $f(x)$
 - or by the cumulative distribution function (cdf) $F(t) \stackrel{\text{def}}{=} \text{Prob}(x \leq t)$.
- If we know the pdf $f(x)$, then we can reconstruct the cdf as $F(t) = \int_{-\infty}^t f(x) dx$.
- Vice versa, if we know the cdf $F(t)$, we can reconstruct the pdf as its derivative $f(x) = F'(x)$.

5. How to Make Decisions Under Probabilistic Uncertainty: A Theoretical Recommendation

- Let us assume that we have several possible decisions whose outcomes are characterized by the pdfs $f_i(x)$.
- Decisions of a rational person can be characterized by a function $u(x)$ called *utility function*.
- Namely, a rational person should select a decision with the largest $\int f_i(x) \cdot u(x) dx$.
- A decision corresponding to pdf $f_1(x)$ is preferable to the decision corresponding to pdf $f_2(x)$ if

$$\int f_1(x) \cdot u(x) dx > \int f_2(x) \cdot u(x) dx.$$

- This condition is equivalent to $\int \Delta f(x) \cdot u(x) dx > 0$, where $\Delta f(x) \stackrel{\text{def}}{=} f_1(x) - f_2(x)$.

6. From a Theoretical Recommendation to Practical Decision

- Theoretically, we can determine the utility function of the decision maker.
- However, since a determination is very time-consuming.
- So, it is rarely done in real financial situations.
- As a result, in practice, we only have a partial information about the utility function.
- One thing we know for sure if that:
 - the larger the monetary gain x ,
 - the better the resulting situation.
- So, the utility function $u(x)$ is increasing.

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7. How to Make Decisions When We Only Know that Utility Function Is Increasing

- The first decision is better if $I \stackrel{\text{def}}{=} \int \Delta f(x) \cdot u(x) dx \geq 0$, where $\Delta f(x) = f_1(x) - f_2(x)$.
- When is the integral I non-negative?
- In reality, both gains and losses are bounded by some value T , so $\int \Delta f(x) \cdot u(x) dx = \int_{-T}^T \Delta f(x) \cdot u(x) dx$.
- Integrating by parts, we get $I = - \int \Delta F(x) \cdot u'(x)$, where $\Delta F(x) \stackrel{\text{def}}{=} F_1(x) - F_2(x)$.
- Here, $u'(x) \geq 0$.
- Thus, if $\Delta F(x) \leq 0$ for all x , i.e., if $F_1(x) \leq F_2(x)$ for all x , then $I \geq 0$ and so, the 1st decision is better.
- This is the main idea behind *stochastic dominance*.

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8. Stochastic Dominance: Discussion

- We showed: the condition $F_1(x) \leq F_2(x)$ for all x is sufficient to conclude that the first alternative is better.
- This condition is also necessary: if $F_1(x_0) > F_2(x_0)$ for some x_0 , then the 2nd decision is better for some $u(x)$.
- Sometimes, we have additional information about the utility function.
- E.g., the same amount of extra money h is more valuable for a poor person than for the rich person:

$$\text{if } x < y \text{ then } u(x + h) - u(x) \geq u(y + h) - u(y).$$

- This condition is equivalent to convexity of $u(x)$.
- For convex $u(x)$, we can perform one more integration by parts and get an even more powerful criterion.

9. What if the Stochastic Dominance Condition Is Satisfied “Almost Always”

- Let us return to the simple situation when we only know that utility is increasing, i.e., that $u'(x) \geq 0$.
- In this case, if we know that $F_1(x) \leq F_2(x)$ for *all* x , then the first alternative is better.
- In many cases, we can use this criterion.
- Sometimes, the inequality $F_1(x) \leq F_2(x)$ holds for *most* values x – except for some small interval.
- It would be nice to be able to make decisions even if we have *approximate* stochastic dominance.
- We show that, under reasonable assumptions,
 - we can make definite decisions
 - even under such approximate stochastic dominance.

10. Additional Reasonable Assumptions about $u(x)$

- In theory, we can have $u'(x) = 10^{-40}$ or $u'(x) = 10^{40}$.
- From the economical viewpoint, however, such too small or too large numbers make no sense.
- If the derivative is too small, this means that the person does not care whether he/she gets more money.
- This may be true for a monk leading a spiritual life, but not for agents who look for profit.
- If $u'(x)$ is too large, this means that adding a very small amount of money leads to a drastic increase in utility.
- This is usually not the case.
- Thus, the derivative $u'(x)$ should not too small or too large: there should be some values $0 < s < L$ s.t.:

$$s \leq u'(x) \leq L \text{ for all } x.$$

11. The Assumption $0 < s \leq u'(x) \leq L$ Helps Us Deal With Approximate Stochastic Dominance

- We want to make sure that $J \stackrel{\text{def}}{=} \int \Delta F(x) \cdot u'(x) dx \leq 0$.
- Let us use the fact that

$$J = \int_{x:\Delta F(x) \leq 0} \Delta F(x) \cdot u'(x) dx + \int_{x:\Delta F(x) > 0} \Delta F(x) \cdot u'(x) dx.$$

- Here, $\int_{x:\Delta F(x) \leq 0} \Delta F(x) \cdot u'(x) dx \leq -s \cdot \int_{x:\Delta F(x) \leq 0} |\Delta F(x)| dx$.
- Also, $\int_{x:\Delta F(x) > 0} \Delta F(x) \cdot u'(x) dx \leq L \cdot \int_{x:\Delta F(x) > 0} \Delta F(x) dx$.
- Thus, $J \leq -s \cdot \int_{x:\Delta F(x) \leq 0} |\Delta F(x)| dx + L \cdot \int_{x:\Delta F(x) > 0} \Delta F(x) dx$.
- J is non-negative when the bound is non-negative, i.e., when

$$\int_{x:\Delta F(x) > 0} \Delta F(x) dx \leq \frac{s}{L} \cdot \int_{x:\Delta F(x) \leq 0} |\Delta F(x)| dx.$$

- This condition is satisfied when the set of all values x for which $\Delta F(x) > 0$ is small.

12. How To Make Decisions Under Approximate Stochastic Dominance: Main Result

- We have two alternatives A_1 and A_2 , characterized by the cumulative distribution functions $F_1(x)$ and $F_2(x)$.
- We need to decide which alternative is better.
- The utility function $u(x)$ describing the agent's attitude to monetary values x is non-decreasing: $u'(x) \geq 0$.
- We know two positive numbers $s < L$ such that for every x , we have $s \leq u'(x) \leq L$.
- *Known:* if $\Delta F(x) \stackrel{\text{def}}{=} F_1(x) - F_2(x) \leq 0$ for all x , then A_1 is better.
- *New:* A_1 is still better if $\Delta F(x) > 0$ for some x , but the set of all such x is small, in the sense that

$$\int_{x:\Delta F(x)>0} \Delta F(x) dx \leq \frac{s}{L} \cdot \int_{x:\Delta F(x)\leq 0} |\Delta F(x)| dx.$$

$$13. \int_{x:\Delta F(x)>0} \Delta F(x) dx \leq \frac{s}{L} \cdot \int_{x:\Delta F(x)\leq 0} |\Delta F(x)| dx :$$

Discussion

- Adding $\int_{x:\Delta F(x)\leq 0} |\Delta F(x)| dx$ to both sides of our inequality and denoting $f_+(x) \stackrel{\text{def}}{=} \max(f(x), 0)$, we get:

$$\int |F_2(x) - F_1(x)| dx \leq \left(1 + \frac{s}{L}\right) \cdot \int (F_2(x) - F_1(x))_+ dx, \text{ i.e.,}$$

$$P \stackrel{\text{def}}{=} \frac{\int (F_2(x) - F_1(x))_+ dx}{\int |F_2(x) - F_1(x)| dx} \geq \frac{1}{1 + \frac{s}{L}}.$$

- The above ratio P :
 - is called the *Proportional Expected Difference*,
 - is used in the analysis of transitivity of stochastic relations.
- *Comment:* a similar idea can be used when $u''(x) \leq 0$ for “almost all” x .

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