Across-the-Board Spending Cuts Are Very Inefficient: A Proof

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1. Across-the-Board Spending Cuts Are Ubiquitous

- Sometimes, a department or a country faces an unexpected decrease in funding.
- In this case, it is necessary to balance the budget by making some spending cuts.
- Usually, an across-the board cut is performed: all the spending items are decreased by the same percentage.
- For example, all the salaries are decreased by the same percentage.
- These cuts apply to everyone on the same basis.
- Thus, they are perceived as fair.
- This explains the ubiquity of across-the-board cuts.

2. Across-the-Board Cuts May Sound Fair, But Are They Economically Efficient?

- The fact that such cuts are fair do not necessarily mean that they are economically efficient.
- If we divide all the wealth equally between everyone, this may be a very fair division.
- However, because of its lack of motivations to work harder, this will not be an economically efficient idea.
- The current impression is that across-the-board cuts are not optimal, but they are economically reasonable.
- Our quantitative analysis shows that their economic effect is much worse than it is usually perceived.
- To make our argument convincing, we make the mathematical arguments simple.

3. Let Us Formulate The Problem in Precise Terms

- Let x be the original budget, with amounts $x_1, \ldots, x_n$ allocated to different categories: $\sum\limits_{i=1}^{n} x_i = x$.
- If we only have amount $y < x$, we need to select values $y_i \le x_i$ for which $\sum\limits_{i=1}^{n} y_i = y$.
- Across-the-board cut means selecting $y_i = (1 - \delta) \cdot x_i$, where $\delta = 1 - \dfrac{y}{x}$.
- Let $f(y_1, \ldots, y_n)$ be the objective function describing our preferences.
- We should thus select y_i for which $f(y_1, \ldots, y_n) \to \max$ under the constraint $\sum\limits_{i=1}^{n} y_i = y$.

4. Possibility of Linearization

- Usually, the relative size of the overall cut $\Delta y \stackrel{\text{def}}{=} x - y$ does not exceed 10%.
- By economic standards, a 10% cut is huge.
- However, from the mathematical viewpoint, it is *small*: terms quadratic in this cut can be safely ignored:

$$(10\%)^2 = 0.1^2 = 0.01 = 1\% \ll 10\%.$$

- If we ignore terms quadratic in $\Delta y_i \stackrel{\text{def}}{=} x_i - y_i$ in Taylor series, we get $f(y_1, \ldots, y_n) = f_x - \sum\limits_{i=1}^n c_i \cdot \Delta y_i$, where:

$$f_x \stackrel{\text{def}}{=} f(x_1, \ldots, x_n) \text{ and } c_i \stackrel{\text{def}}{=} \frac{\partial f}{\partial y_i} \geq 0.$$

- Maximizing $f(y_1, \ldots, y_n)$ means minimizing the difference $f_x - f(y_1, \ldots, y_n) = \sum\limits_{i=1}^n c_i \cdot \Delta y_i$.

5. Optimal vs. Across-the-Board Spending Cuts

- We need to minimize the sum $\sum\limits_{i=1}^n c_i \cdot \Delta y_i$ under the constraint $\sum\limits_{i=1}^n \Delta y_i = \Delta y$.
- One can see that min is attained when all the cuts are allocated to the category i with the smallest c_i:

$$c_i = \min_j c_j, \text{ so } f_x - f_y = \left(\min_i c_i\right) \cdot \Delta y.$$

- For across-the-board cuts, $\Delta y_i = \delta \cdot x_i$, so the loss is equal to $f_x - f_\delta = \Delta y \cdot \sum\limits_{i=1}^n c_i \cdot \delta x_i$, where $\delta x_i \stackrel{\text{def}}{=} \dfrac{x_i}{x}$.
- The values of c_i and Δx_i depend on many factors which we do not know beforehand.
- So it makes sense to treat c_i and Δx_i as random variables.

6. Treating c_i and δx_i as Random Variables

- A reasonable way to compare two random variables is to compare their mean values.
- This is what we mean, e.g., when we say that Swedes are, on average taller than Americans: that
  - the average height of a Swede is larger than
  - the average height of an American.
- We have no reason to believe that the variables c_i corresponding to different budget items are correlated.
- Same with δx_j.
- So, it makes sense to assume that all these variables are independent.

7. This Is In Line with the Maximum Entropy Approach

- According to this approach:
  - if there are several possible probability distributions consistent with our knowledge,
  - it makes sense to select the one which has the largest uncertainty (entropy)

$$S \stackrel{\text{def}}{=} -\int \rho(x) \cdot \ln(\rho(x))\, dx.$$

- In particular:
  - when we only know the marginal distributions, with pdf $\rho_1(x_1)$ and $\rho_2(x_2)$,
  - the Maximum Entropy approach selects

$$\rho(x_1, x_2) = \rho_1(x_1) \cdot \rho_2(x_2);$$

  - this corresponds to independence.

8. Consequence of Independence

- From $f_x - f_\delta = \Delta y \cdot \sum\limits_{i=1}^{n} c_i \cdot \delta x_i$, we conclude that

$$E[f_x - f_\delta] = \Delta y \cdot \sum_{i=1}^{n} E[c_i] \cdot E[\delta x_i].$$

- We have no reason to believe that some values δx_i are larger.
- So it makes sense to assume that they have the same value of $E[\delta x_i]$.
- From $\sum\limits_{i=1}^{n} \delta x_i = 1$, we get $\sum\limits_{i=1}^{n} E[\delta x_i] = 1$, hence

$$E[\delta x_i] = \frac{1}{n}, \text{ and } E[f_x - f_\delta] = \Delta y \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} E[c_i].$$

- The optimal value is $E[f_x - f_y] = \Delta y \cdot E\left[\min\limits_i c_i\right]$.

9. Analysis of the Problem

- To compare $E[f_x - f_\delta]$ and $E[f_x - f_y]$, we need to select probability distributions for c_i.
- We do not have much information about the values c_i.
- We know that $c_i \geq 0$. We also know that these values cannot be too large.
- Thus, we usually know an upper bound c on these values.
- So, for each i, the only information that we have about c_i is that it is located on the interval $[0, c]$.
- Under this information, the Maximum Entropy approach recommends the uniform distribution.

10. Conclusion

- Uniform distributions are in perfect accordance with common sense:
  - if we have no reason to believe that some values from this interval are more probable,
  - then it is reasonable to assume that all these values have the exact same probability,
  - i.e., that the distribution is indeed unform.
- For uniform distribution, $E[f_x - f_\delta] = \dfrac{1}{2} \cdot \Delta y \cdot c$ and $E[f_x - f_\delta] = \dfrac{1}{n+1} \cdot \Delta y \cdot c.$
- So, the average decrease caused by the across-the-board cuts is $\dfrac{n+1}{2}$ larger than the optimal value.
- Thus, on average, across-the-board cuts are indeed very inefficient.

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12. Appendix: Derivations

- Let us find the probability distribution for the minimum $m \stackrel{\text{def}}{=} \min\limits_i c_i$.
- For each v,we have

$$m \geq v \Leftrightarrow (c_1 > v)\,\&\,\ldots\,\&\,(c_n > v), \text{ so}$$

$$\text{Prob}(m > v) = \text{Prob}((c_1 > v)\,\&\,\ldots\,\&\,(c_n > v)).$$

- Since $c_1, \ldots, c_n$ are all independent, we have

$$\text{Prob}(m > v) = \text{Prob}(c_1 > v) \cdot \ldots \cdot \text{Prob}(c_n > v).$$

- For each i, c_i is uniform, so $\text{Prob}(c_i > v) = \dfrac{c - v}{c}$, and

$$\text{Prob}(m > v) = \left(\frac{c-v}{c}\right)^n, \text{ and}$$

$$F_m(v) = \text{Prob}(m \leq v) = 1 - \text{Prob}(m > v) = 1 - \left(\frac{c-v}{c}\right)^n.$$

13. Derivations (cont-d)

- Thus, $\rho_m(v) = \dfrac{dF_m(v)}{dv} = \dfrac{n}{c^n} \cdot (c-v)^{n-1}$.
- So, $E[m] = \int_0^c v \cdot \rho_m(v)\,dv = \int_0^c v \cdot \dfrac{n}{c^n} \cdot (c-v)^{n-1}\,dv$.
- For $w \stackrel{\text{def}}{=} c - v$, we have $v = c - w$, $dv = -dw$, and

$$E[m] = \frac{n}{c^n} \cdot \int_0^c (c-w) \cdot w^{n-1}\,dw =$$

$$\frac{n}{c^n} \cdot \left(c \cdot \int_0^c w^{n-1}\,dw - \int_0^c w^n\,dw\right) = \frac{n}{c^n} \cdot \left(c \cdot \frac{c^n}{n} - \frac{c^{n+1}}{n+1}\right) =$$

$$\frac{n}{c^n} \cdot c^{n+1} \cdot \left(\frac{1}{n} - \frac{1}{n+1}\right) = c \cdot n \cdot \frac{1}{n \cdot (n+1)} = \frac{c}{n+1}.$$

- Thus, $E[f_x - f_\delta] = \Delta y \cdot E\left[\min_i c_i\right] = \dfrac{1}{n+1} \cdot \Delta y \cdot c$.