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Invariance Explains Multiplicative and Exponential Skedactic Functions

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1. Linear Dependencies Are Ubiquitous

- In many practical situations, a quantity y depends on several other quantities $x_1, \ldots, x_n$: $y = f(x_1, \ldots, x_n)$.

- Often, the ranges of x_i are narrow: $x_i \approx x_i^{(0)}$ for some $x_i^{(0)}$, so $\Delta x_i \stackrel{\text{def}}{=} x_i - x_i^{(0)}$ are relatively small.

- Then, we can expand the dependence of y on $x_i = x_i^{(0)} + \Delta x_i$ in Taylor series and keep only linear terms:

$$y = f(x_1^{(0)} + \Delta x_1, \ldots, x_n^{(0)} + \Delta x_n) \approx a_0 + \sum_{i=1}^{n} a_i \cdot \Delta x_i,$$

$$\text{where } a_0 \stackrel{\text{def}}{=} f\left(x_1^{(0)}, \ldots, x_n^{(0)}\right) \text{ and } a_i \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_i}.$$

- Substituting $\Delta x_i = x_i - x_i^{(0)}$ into this formula, we get $y \approx c + \sum_{i=1}^{n} a_i \cdot x_i$, where $c \stackrel{\text{def}}{=} a_0 - \sum_{i=1}^{n} a_i \cdot x_i^{(0)}$.

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2. Linear Dependencies Are Approximate

- Usually,
  - in addition to the quantities $x_1, \ldots, x_n$ that provide the most influence on y,
  - there are also many other quantities that (slightly) influence y,
  - so many that it is not possible to take all of them into account.

- Since we do not take these auxiliary quantities into account, the linear dependence is approximate.

- The approximation errors $\varepsilon \stackrel{\text{def}}{=} y - \left(c + \sum_{i=1}^{n} a_i \cdot x_i \right)$ depend on un-observed quantities.

- So, we cannot predict ε based only on the observed quantities $x_1, \ldots, x_n$.

- It is therefore reasonable to view ε as random variables.

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3. Skedactic Functions

- A natural way to describe a random variable is by its moments.

- If the first moment is not 0, then we can correct this bias by appropriately updating the constant c.

- Since the mean is 0, the second moment coincides with the variance v.

- The dependence $v(x_1, \ldots, x_n)$ is known as the *skedactic function*.

- In econometric applications, two major classes of skedactic functions have been empirically successful:

  - multiplicative $v(x_1, \ldots, x_n) = c \cdot \prod_{i=1}^{n} |x_i|^{\gamma_i}$ and

  - exponential $v(x_1, \ldots, x_n) = \exp\left(\alpha + \sum_{i=1}^{n} \gamma_i \cdot x_i\right)$.

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4.   Problems and What We Do

- *Problems:*
    - Neither of the empirically successful skedactic functions has a theoretical justification.
    - In most situations, the multiplication function results in more accurate estimates.
    - This fact also does not have an explanation.

- *What we do:* we use invariance ideas to:
    - explain the empirical success of multiplicative and exponential skedactic functions, and
    - come up with a more general class of skedactic functions.

5.   Natural Invariance: Scaling

- Many economics quantities correspond to prices, wages, etc. and are thus expressed in terms of money.

- The numerical value of such a quantity depends on the choice of a monetary unit.

- For example, when a European country switches to Euro from its original currency,

    – the actual incomes do not change, but
    – all the prices and wages get multiplied by the corresponding exchange rate k: $x_i \to x_i' = k \cdot x_i$.

- Similarly, the numerical amount (of oil or sugar), changes when we change units.

- For example, for oil, we can use barrels or tons.

- When the numerical value of a quantity is multiplied by k, its variance gets multiplied by k^2.

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6.   Scaling (cont-d)

- Changing the measuring units for $x_1, \ldots, x_n$ does not change the economic situations.

- So, it makes sense to require that the skedactic function also does not change under such re-scaling: namely,

  – for each combination of re-scalings on inputs,

  – there should be an appropriate re-scaling of the output after which the dependence remains the same.

- In precise terms, this means that:

  – for every combination of numbers $k_1, \ldots, k_n$,

  – there should exist a value $k = k(k_1, \ldots, k_n)$ with the following property:

  $v = v(x_1, \ldots, x_n)$ if and only if $v' = v(x'_1, \ldots, x'_n)$,
  where $v' = k \cdot v$ and $x'_i = k_i \cdot x_i$.

7.   Shift and Shift-Invariance

- While most economic quantities are scale-invariant, some are not.

- For example, the unemployment rate is measured in percents, there is a fixed unit.

- Many such quantities can have different numerical values depending on how we define a starting point.

- For example, we can measure unemployment:
  - either by the usual percentage x_i, or
  - by the difference $x_i - k_i$, where $k_i > 0$ is what economists mean by full employment.

- It is thus reasonable to consider *shift-invariant* skedactic functions:

$$\forall k_1, \ldots, k_n \, \exists k \, (v = v(x_1, \ldots, x_n) \Leftrightarrow v' = f(x'_1, \ldots, x'_n)),$$
$$\text{where } v' = k \cdot v \text{ and } x'_i = x_i + k_i.$$

8.   Scale Invariance: Main Result

- **Definition.** We say that a non-negative measurable function $v(x_1, \ldots, x_n)$ is *scale-invariant* if:

  - for every n-tuple of real numbers $(k_1, \ldots, x_n)$,
  - there exists a real number $k = k(k_1, \ldots, k_n)$ for which, for every $x_1, \ldots, x_n$ and v:

  $$v = v(x_1, \ldots, x_n) \Leftrightarrow v' = v(x'_1, \ldots, x'_n), \text{ where}$$
  $$v' = k \cdot v \text{ and } x'_i = k_i \cdot x_i.$$

- **Proposition.** *A skedactic function is scale-invariant if and only it has the form*

  $$v(x_1, \ldots, x_n) = c \cdot \prod_{i=1}^{n} |x_i|^{\gamma_i} \text{ for some } c \text{ and } \gamma_i.$$

- **Discussion.** Thus, scale-invariance explains the use of multiplicative skedactic functions.

9. Shift Invariance: Main Result

- **Definition.** We say that a non-negative measurable function $v(x_1, \ldots, x_n)$ is *shift-invariant* if:

  - for every n-tuple of real numbers $(k_1, \ldots, k_n)$,
  - there exists a real number $k = k(k_1, \ldots, k_n)$ for which, for every $x_1, \ldots, x_n$ and v:
    $$v = v(x_1, \ldots, x_n) \Leftrightarrow v' = v(x'_1, \ldots, x'_n), \text{ where}$$
    $$v' = k \cdot v \text{ and } x'_i = x_i + k_i.$$

- **Result.** *A skedactic function is shift-invariant* $\Leftrightarrow$

  $$v(x_1, \ldots, x_n) = \exp\left(\alpha + \prod_{i=1}^{n} \gamma_i \cdot x_i\right) \text{ for some } \alpha \text{ and } \gamma_i.$$

- **Discussion.** Thus, shift-invariance explains the use of exponential skedactic functions.

- It also explains why multiplicative functions are more often useful: scaling is ubiquitous, shift is rarer.

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10. General Case: Some Inputs Are Scale-Invariant and Some Are Shift-Invariant

- **Definition.** We say that a non-negative measurable function $v(x_1, \ldots, x_n)$ is *m-invariant* if:

  - for every n-tuple of real numbers $(k_1, \ldots, k_n)$,
  - there exists a real number $k = k(k_1, \ldots, k_n)$ for which, for every $x_1, \ldots, x_n$ and v:

  $v = v(x_1, \ldots, x_n) \Leftrightarrow v' = v(x'_1, \ldots, x'_n)$, where
  $v' = k \cdot v$, $x'_i = k_i \cdot x_i$ for $i \leq m$, $x'_i = x_i + k_i$ for $i > m$.

- **Result.** *A skedactic function is m-invariant* $\Leftrightarrow$

  $$v(x_1, \ldots, x_n) = \exp\left(\alpha + \sum_{i=1}^{m} \gamma_i \cdot \ln(|x_i|) + \sum_{i=m+1}^{n} \gamma_i \cdot x_i\right).$$

- For $m = n$, we get multiplicative skedactic function, for $m = 0$, we get the exponential one.

- For other m, we get new possibly useful expressions.

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12. Proof of the First Result

- It is easy to check that the multiplicative skedactic function is scale-invariant: take $k = \prod\limits_{i=1}^{n} |k_i|^{\gamma_i}$.

- Vice versa, the equivalence condition means that

$$k(k_1, \ldots, k_n) \cdot v(x_1, \ldots, x_n) = v(k_1 \cdot x_1, \ldots, k_n \cdot x_n).$$

- Thus, $k(k_1, \ldots, k_n) = \dfrac{v(k_1 \cdot x_1, \ldots, k_n \cdot x_n)}{v(x_1, \ldots, x_n)}$ is a ratio of measurable functions hence measurable.

- Let us consider two tuples $(k_1, \ldots, k_n)$, $(k'_1, \ldots, k'_n)$.

- If we first use the first re-scaling, i.e., go from x_i to $x'_i = k_i \cdot x_i$, we get

$$v(x'_1, \ldots, x'_n) = k(k_1, \ldots, k_n) \cdot v(x_1, \ldots, x_n).$$

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13. Proof (cont-d)

- If we then apply, to the new values x_i', an additional re-scaling $x_i' \to x_i'' = k_i' \cdot x_i'$, we similarly conclude that

$$v(x_1'', \ldots, x_n'') = k(k_1', \ldots, k_n') \cdot v(x_1', \ldots, x_n') =$$
$$k(k_1', \ldots, k_n') \cdot k(k_1, \ldots, k_n) \cdot v(x_1, \ldots, x_n).$$

- We could also get the values x_i'' if we directly multiply each value x_i by the product $k_i'' \stackrel{\text{def}}{=} k_i' \cdot k_i$, thus

$$v(x_1'', \ldots, x_n'') = k(k_1' \cdot k_1, \ldots, k_n' \cdot k_n) \cdot v(x_1, \ldots, x_n).$$

- Thus, $k(k_1' \cdot k_1, \ldots, k_n' \cdot k_n) \cdot v(x_1, \ldots, x_n) = k(k_1', \ldots, k_n') \cdot k(k_1, \ldots, k_n) \cdot v(x_1, \ldots, x_n).$

- If the skedactic function is always equal to 0, then it is multiplicative, with $c = 0$.

- If it is not everywhere 0, this means that its value is different from 0 for some combination of values $x_1, \ldots, x_n$.

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14. Proof (cont-d)

- For this tuple, we get $k(k'_1 \cdot k_1, \ldots, k'_n \cdot k_n) = k(k'_1, \ldots, k'_n) \cdot k(k_1, \ldots, k_n)$.

- When $k_i = k'_i = -1$ for some i and $k'_j = k_j = 1$ for all $j \neq i$, i, we get $1 = k(1, \ldots, 1) = k^2(k_1, \ldots, k_n)$.

- Since the function k_i is non-negative, this means that $k(k_1, \ldots, k_n) = 1$.

- Thus, the value $k(k_1, \ldots, k_n)$ does not change if we change the signs of k_i: $k(k_1, \ldots, k_n) = k(|k_1|, \ldots, |k_n|)$.

- Thus, $\ln(k(k'_1 \cdot k_1, \ldots, k'_n \cdot k_n)) = \ln(k(k'_1, \ldots, k'_n)) + \ln(k(k_1, \ldots, k_n))$.

- For an auxiliary function $K(K_1, \ldots, K_n) \overset{\text{def}}{=} \ln(k(\exp(K_1), \ldots, \exp(K_n)))$, we thus get

$$K(K'_1 + K_1, \ldots, K'_n + K_n) = K(K'_1, \ldots, K'_n) + K(K_1, \ldots, K_n).$$

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15.   Proof: Conclusion

- We know that

$$K(K'_1 + K_1, \ldots, K'_n + K_n) = K(K'_1, \ldots, K'_n) + K(K_1, \ldots, K_n).$$

- It is known that measurable functions with this property are linear: $K(K_1, \ldots, K_n) = \sum_{i=1}^{n} \gamma_i \cdot K_i$, so

$$k(k_1, \ldots, k_n) = \exp\left(\sum_{i=1}^{n} \gamma_i \cdot \ln(|k_i|)\right) = \prod_{i=1}^{n} |k_i|^{\gamma_i}.$$

- We also have $v(x_1, \ldots, x_n) = k(x_1, \ldots, x_n) \cdot v(1, \ldots, 1).$

- Thus, we get the desired formula for the multiplicative skedastic function $v(x_1, \ldots, x_n) = c \cdot \prod_{i=1}^{n} |x_i|^{\gamma_i}.$

- Similar proofs hold for shift-invariance and for the general case.