Econometric Models of Probabilistic Choice: Beyond McFadden's Formulas

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1. Traditional (Deterministic Choice) Approach to Decision Making

- In the traditional approach to decision making, we assume that for every two alternatives a and b:
  - either the decision maker always prefers a,
  - or the decision maker always prefers b,
  - or, to the decision maker, a and b are equivalent.
- Then, decision maker's preferences can be described by *utilities* defined as follows.
- We select two alternatives which are not present in the original choices:
  - a very bad alternative a_0, and
  - a very good alternative a_1.
- Then, each actual alternative a is better than a_0 and worse than a_1: $a_0 < a < a_1$.

2. Traditional Decision Making (cont-d)

- To gauge the quality of the alternative a to the decision maker, we can consider lotteries $L(p)$ in which:
  - we get a_1 with probability p and
  - we get a_0 with the remaining probability $1 - p$.
- For every p, we either have $L(p) < a$ or $a < L(p)$ or $L(p) \sim a$.
- When $p = 1$, $L(1) = a_1$, thus $a < L(1)$.
- When $p = 0$, $L(0) = a_0$, thus $L(0) < a$.
- Clearly, the larger the probability p of the very good outcome, the better the lottery; thus, if $p < p'$, then:
  - $a < L(p)$ implies $a < L(p')$, and
  - $L(p') < a$ implies $L(p) < a$.

3. Traditional Decision Making (cont-d)

- Therefore, we can conclude that

$$\sup\{p : L(p) < a\} = \inf\{p : a < L(p)\}.$$

- $u(a) \stackrel{\text{def}}{=} \sup\{p : L(p) < a\} = \inf\{p : a < L(p)\}$ has the following properties:
    - if $p < u$, then $L(p) < a$; and
    - if $p > a$, then $a < L(p)$.
- In particular, for every small $\varepsilon > 0$, we have

$$L(u(a) - \varepsilon) < a < L(u(a) + \varepsilon).$$

- So, a is "equivalent" to the lottery $L(p)$ in which a_1 is selected with the probability $p = u(a)$: $a \equiv L(u(a))$.
- This probability $u(a)$ is what is known as *utility*.
- Once we know all the utility values, we select the alternative with the largest utility.

4. Traditional Decision Making (final)

- Indeed, as we have mentioned, $p < p'$ implies that $L(p) < L(p')$, so when $u(a) < u(b)$, we have

$$a \equiv L(u(a)) < L(u(b)) \equiv b \text{ and thus } a < b.$$

- The above definition of utility depends on the choice of two alternatives a_0 and a_1.
- If we select different a'_0 and a'_1, then, as one can show, we get $u'(a) = k \cdot u(a) + \ell$ for some $k > 0$ and ℓ.
- Thus, *utility is defined modulo linear transformation.*

5. Actual Choices Are Often Probabilistic

- People sometimes make different choices when repeatedly presented with the same alternatives a and b.
- This is especially true when the compared alternatives a and b are close in value.
- In such situations, we would like to predict the probability $P(a, A)$ of a from a set A.
- We can still have a deterministic distinction: $b > a$ if the person selects a more frequently than b:

$$P(a, \{a, b\}) > 0.5.$$

- Based on $>$, we can determine the utilities $u(a)$.
- It is reasonable to assume that $P(a, A)$ depends only on the utilities $u(a)$, $\ldots$, $u(b)$.

6. McFadden's Formulas for Probabilistic Selection

- The 2001 Nobelist D. McFadden proposed the following formula for the desired probability $P(a, A)$:

$$P(a, A) = \frac{\exp(\beta \cdot u(a))}{\sum\limits_{b \in A} \exp(\beta \cdot u(b))}.$$

- In many practical situations, this formula indeed describes people's choices really well.
- In some case, alternative formulas provide a better explanation of the empirical choices.
- In this talk, we use natural symmetries to come up with an appropriate generalization of McFadden's formulas.

7. Analysis of the Problem

- We may have many different alternatives a, b, ...
- In some cases, we prefer a, in other cases, we prefer b.
- It is reasonable to require that:
  - once we have decided on selecting either a or b, then
  - the relative frequency of selecting a should be the same as when we simply select between a and b:

$$\frac{P(a,A)}{P(b,A)} = \frac{P(a,\{a,b\})}{P(b,\{a,b\})} = \frac{P(a,\{a,b\})}{1 - P(a,\{a,b\})}.$$

- Let us add a new alternative a_n to our list, then:

$$\frac{P(a,A)}{P(a_n,A)} = \frac{P(a,\{a,a_n\})}{1 - P(a,\{a,a_n\})}, \text{ so } P(a,A) = P(a_n,A)\cdot f(a),$$

$$\text{where } c \stackrel{\text{def}}{=} P(a_n,A) \text{ and } f(a) \stackrel{\text{def}}{=} \frac{P(a,\{a,a_n\})}{1 - P(a,\{a,a_n\})}.$$

8. Analysis of the Problem (cont-d)

- c can be found from the condition that one of $b \in A$ will be selected: $\sum\limits_{b \in A} P(b, A) = 1$, so $P(a, A) = \dfrac{f(a)}{\sum\limits_{b \in A} f(b)}$.

- We assumed that the probabilities depend only on the utilities $u(a)$.

- We thus conclude that $f(a)$ must depend only on the utilities: $f(a) = F(u(a))$ for some $F(u)$, and

$$P(a, A) = \frac{F(u(a))}{\sum\limits_{b \in A} F(u(b))}.$$

- Thus, all we need is to find an appropriate function $F(u)$.

9. Properties of $F(u)$

- The better the alternative a, i.e., the larger $u(a)$, the higher should be the probability $P(a, A)$.
- Thus, $F(u)$ is an increasing function of the utility u.
- If we multiply all the values of $F(u)$ by a constant, we will get the exact same probabilities.
- Utilities are defined modulo a general linear transformation.
- In particular, it is possible to add a constant to all the utility values $u(a) \to u'(a) = u(a) + c$.
- Since this shift does not change the preferences, it is reasonable to require that for $u'(a)$, we get the same probabilities.
- Using new utility values $u'(a) = u(a) + c$ means that we replace $F(u(a))$ with $F(u'(a)) = F(u(a) + c)$.

10. Deriving McFadden's Formula

- Using new utility values $u'(a) = u(a) + c$ means that we replace $F(u(a))$ with $F(u'(a)) = F(u(a) + c)$.
- This is equivalent to using the *original* utility values but with a *new function* $F'(u) \stackrel{\text{def}}{=} F(u + c)$.
- The functions $F(u)$ and $F'(u)$ describe the same probabilities if and only if $F'(u) = C \cdot F(u)$ for some C.
- So, $F(u + c) = C(c) \cdot F(u)$ for some $C(c)$.
- It is known that every monotonic solution to this function equation has the form $F(u) = C_0 \cdot \exp(\beta \cdot u)$.
- This is exactly McFadden's formula.

11. Discussion

- The proof of the function-equation result is somewhat complicated.
- However, under a natural assumption that $F(u)$ is differentiable, this result can be proven rather easily.
- $C(u)$ is a ratio of two differentiable functions $F(u+c)$ and $F(u)$, and is, thus, differentiable.
- Since $F(u)$ and $C(c)$ are differentiable, we can differentiate both sides of the equality by c and take $c=0$.
- As a result, we get $\dfrac{dF}{du} = \beta \cdot F$, where $\beta \stackrel{\text{def}}{=} \dfrac{dC}{dc}_{|c=0}$.
- By moving all the terms with F to one side and all others to the other side, we get: $\dfrac{dF}{F} = \beta \cdot du$.
- Integrating both sides, we get $\ln(F) = \beta \cdot u + C_1$, so $F(u) = \exp(\ln(F)) = C_0 \cdot \exp(\beta \cdot u)$, w/$C_0 \stackrel{\text{def}}{=} e^{C_1}$.

12. Our Main Idea

- Multiplying all the utility values by a constant is also a legitimate transformation for utilities.
- However, this *does* change McFadden's probabilities.
- So, we cannot require that the probability formula not change for *all* possible linear transformations of utility:
  - once we require shift-invariance,
  - we get McFadden's formula
  - which is not scale-invariant.
- So, we should require invariance with respect to *some* family of re-scalings.

13. Main Idea (cont-d)

- If a formula does not change when we apply each transformation, it will also not change:
  - if we apply them one after another,
  - i.e., if we consider a composition of transformations.
- Each shift can be represented as a superposition of many small (infinitesimal) shifts $u \to u + B \cdot dt$.
- Similarly, each scaling can be represented as a superposition of many small scalings $u \to (1 + A \cdot dt) \cdot u$.
- Thus, it is sufficient to consider invariance with respect to an infinitesimal transformation

$$u \to u' = (1 + A \cdot dt) \cdot u + B \cdot dt.$$

14. Main Idea (cont-d)

- Invariance means that the values $F(u')$ lead to the same probabilities as the original values $F(u)$, so:

$$F(u + (A \cdot u + B) \cdot dt) = F(u) + C \cdot F(u) \cdot dt.$$

- Here, by definition of the derivative, $F(u + q \cdot dt) = F(u) + \frac{dF}{du} \cdot q \cdot dt$, so $(A \cdot u + B) \cdot \frac{dF}{du} = C \cdot F(u)$.
- We can separate the variables by moving all the terms with F to one side and all the terms with u to another:

$$\frac{dF}{F} = C \cdot \frac{du}{A \cdot u + B}.$$

- $A = 0$ leads to McFadden's formulas; when $A \neq 0$, then for $x \stackrel{\text{def}}{=} u + \frac{B}{A}$, $\frac{dF}{F} = c \cdot \frac{dx}{x}$, w/$c \stackrel{\text{def}}{=} \frac{C}{A}$.
- Integration leads to $\ln(F) = c \cdot \ln(x) + C_0$, thus $F = C_1 \cdot x^c$ for $C_1 \stackrel{\text{def}}{=} \exp(C_0)$, and $F(u) = C_1 \cdot (u + k)^c$.

15. Conclusions and Discussion

- In addition to the original McFadden's formula, we now have another option $P(a, A) = \dfrac{(u(a)+k)^c}{\sum\limits_{b \in A} (u(b)+k)^c}$.
- This is in good accordance with empirical data.
- This formula is a generalization of McFadden's.
- Indeed, $\exp(\beta \cdot u) = \lim\limits_{n \to \infty} \left(1 + \dfrac{\beta \cdot u}{n}\right)^n$, so for large n, $\exp(\beta \cdot u)$ is indistinguishable from
$$\left(1 + \frac{\beta \cdot u}{n}\right)^n = \left(\frac{\beta}{n}\right)^n \cdot (u+k)^c \text{ for } c = n, k = \frac{n}{\beta}.$$
- So, instead of a 1-parametric McFadden's formula, we now have a 2-parametric formula.
- We can use this additional parameter to get an even more accurate description of the probabilistic choice.

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