How to Explain Ubiquity of Constant Elasticity of Substitution (CES) Production and Utility Functions Without Explicitly Postulating CES

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1. Outline

- The dependence of production on various factors is often described by CES functions.
- These functions are usually explained by postulating two requirements:
  - that the formulas should not change if we change a measuring unit, and
  - a less convincing CES requirement.
- In this paper, we show that the CES requirement can be replaced by a more convincing requirement:
  - that the combined effect of all the factors
  - should not depend on the order in which we combine these factors.

2. CES Production Functions and CES Utility Functions Are Ubiquitous

- Most observed data about production y is well described by the *CES production function*

$$y = \left(\sum_{i=1}^{n} a_i \cdot x_i^r \right)^{1/r}.$$

- Here x_i are the numerical measures of the factors that influence production, such as:
  - amount of capital,
  - amount of labor, etc.
- A similar formula describes how the person's utility y depends on different factors x_i such as:
  - amounts of different types of consumer goods,
  - utilities of other people, etc.

3. How This Ubiquity Is Explained Now

- The current explanation for the empirical success of CES function is based on two requirements.
- The first requirement is that the corresponding function $y = f(x_1, \ldots, x_n)$ is *homogeneous*:

$$f(\lambda \cdot x_1, \ldots, \lambda \cdot x_n) = \lambda \cdot f(x_1, \ldots, x_n).$$

- Meaning: we can describe different factors by using different monetary units.
- The results should not change if we replace the original unit by a one which is λ times smaller.
- After this replacement, the numerical value of each factor changes from x_i to $\lambda \cdot x_i$ and y is replace by $\lambda \cdot y$.
- So, we get exactly the above requirement.

4. Second Requirement

- The second requirement is that $f(x_1, \ldots, x_n)$ should provide *constant elasticity of substitution* (CES).
- The requirement is easier to explain for the case of two factors $n = 2$.
- In this case, this requirement deals with "substitution" situations in which:
  - we change x_1 and then
  - change the original value x_2 to the new value $x_2(x_1)$
  - so that the overall production or utility remain the same.
- The corresponding substitution rate can then be calculated as $s \stackrel{\text{def}}{=} \dfrac{dx_2}{dx_1}$.

5. Second Requirement (cont-d)

- The substitution function $x_2(x_1)$ is explicitly defined by the equation $f(x_1, x_2(x_1)) = \text{const}$, then

$$s = -\frac{f_{,1}(x_1, x_2)}{f_{,2}(x_1, x_2)}, \text{ where } f_{,i}(x_1, x_2) \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_i}(x_1, x_2).$$

- The requirement is that:
  - for each percent of the change in ratio $\frac{x_2}{x_1}$,
  - we get the same constant number of percents change in s: $\frac{ds}{d\left(\frac{x_2}{x_1}\right)} = \text{const}.$
- *Problem:* the CES condition is too mathematical to be convincing for economists.
- *We provide:* more convincing arguments.

6. Main Idea Behind a New Explanation

- In our explanation, we will use the fact that in most practical situations, we combine several factors.
- We can combine these factors in different order. For example:
  - we can first combine the effects of capital and labor into a single characteristic,
  - and then combine it with other factors.
- Alternatively:
  - we can first combine capital with other factors,
  - and only then combine the resulting combined factor with labor, etc.
- The result should not depend on the order in which we perform these combinations.
- We show that this idea implies the CES functions.

7. Derivation of the CES Functions from the Above Idea

- Let us denote a function that combines factors i and j into a single quantity x_{ij} by $f_{i,j}(x_i, x_j)$.
- Similarly, let's denote a function that combines x_{ij} and $x_{k\ell}$ into a single quantity $x_{ijk\ell}$ by $f_{ij,k\ell}(x_{ij}, x_{k\ell})$.
- In these terms, the requirement that the resulting values do not depend on the order means that

$$f_{12,34}(f_{1,2}(x_1, x_2), f_{3,4}(x_3, x_4)) = f_{13,24}(f_{1,3}(x_1, x_3), f_{2,4}(x_2, x_4)).$$

- In both production and utility situations, for each i and j, $f_{i,j}(x_i, x_j)$ is increasing in x_i and x_j.
- It is also reasonable to require that:
  - the function $f_{i,j}(x_i, x_j)$ is continuous, and
  - when one of the factors tends to infinity, the result also tends to infinity.

8. Derivation (cont-d)

- Under these assumptions, $f(a, b)$ is *invertible*:
  - for every $a \in A$ and for every $c \in C$, there exists a unique value $b \in B$ for which $c = f(a, b)$;
  - for every $b \in B$ and for every $c \in C$, there exists a unique value $a \in A$ for which $f(a, b) = c$.
- It is known that:
  - for every set of invertible operations that satisfy the generalized associativity requirement,
  - there exists an Abelian group G and 1-1 mappings $r_i : X_i \to G$, $r_{ij} : X_{ij} \to G$ and $r_X : X \to G$
  - for which, for all $x_i \in X_i$ and $x_{ij} \in X_{ij}$, we have

$$f_{ij}(x_i, x_j) = r_{ij}^{-1}(g(r_i(x_i), r_j(x_j))) \text{ and}$$

$$f_{ij,kl}(x_{ij}, x_{k\ell}) = r_X^{-1}(g(r_{ij}(x_{ij}), r_{k\ell}(x_{k\ell}))).$$

9. Groups and Abelian Groups: Reminder

- A set G with an associative operation $g(a, b)$ and a unit element e ($g(a, e) = g(e, a) = a$) is called a *group*
  - if every element is invertible, i.e.,
  - if for every a, there exists an a' for which

$$g(a, a') = e.$$

- A group in which the operation $g(a, b)$ is commutative is known as *Abelian*.

10. Discussion

- All continuous 1-D Abelian groups with order-preserving operations are isomorphic to $(\mathbb{R}, +)$.
- Here, $(\mathbb{R}, +)$ is the additive group of real numbers, with
$$g(a, b) = a + b.$$
- Thus, we can conclude that all combining operations have the form
$$f_{ij}(x_i, x_j) = r_{ij}^{-1}(r_i(x_i) + r_j(x_j)).$$
- Equivalently, $f_{ij}(x_i, x_j) = y$ means that
$$r_{ij}(y) = r_i(x_i) + r_j(x_j).$$

11. Let Us Use Homogeneity

- We will now prove that homogeneity leads exactly to the desired CES combinations.
- Homogeneity means that if $r_{ij}(y) = r_i(x_i) + r_j(x_j)$, then for every λ: $r_{ij}(\lambda \cdot y) = r_i(\lambda \cdot x_i) + r_j(\lambda \cdot x_j)$.
- For each $x_i' = x_i + \Delta x_i$, let us find $x_j' = x_j + \Delta x_j$ for which $r_i(x_i) + r_j(x_j)$ remains the same (thus, the combined value y remains the same):

$$r_i(x_1') + r_j(x_j') = r_i(x_i + \Delta x_i) + r_j(x_j + \Delta x_j) = r_i(x_i) + r_j(x_j).$$

- For small Δx_i, $\Delta x_j = k \cdot \Delta x_i + o(\Delta x_i)$ for some k.
- Here, $r_i(x_i + \Delta x_i) = r_i(x_i) + r_i'(x_i) \cdot \Delta x_i + o(\Delta x_i)$, where, as usual, f' denotes the derivative.
- Similarly, $r_j(x_j + \Delta x_j) = r_j(x_j + k \cdot \Delta x_i + o(\Delta x_i)) =$

$$r_j(x_j) + k \cdot r_j'(x_j) \cdot \Delta x_i + o(\Delta x_i).$$

12. Let Us Use Homogeneity (cont-d)

- Thus, the above condition takes the form

$$r_i(x_i)+r_j(x_j)+(r_i'(x_i)+k\cdot r_j'(x_j))\cdot\Delta x_i+o(\Delta x_i) = r_i(x_i)+r_j(x_j).$$

- Thus, $r_i'(x_i) + k \cdot r_j'(x_j) = 0$, and $k = -\dfrac{r_i'(x_i)}{r_j'(x_j)}$.

- For re-scaled values, we similarly get $k = -\dfrac{r_i'(\lambda \cdot x_i)}{r_j'(\lambda \cdot x_j)}$, so $\dfrac{r_i'(\lambda \cdot x_i)}{r_i'(x_i)} = \dfrac{r_j'(\lambda \cdot x_j)}{r_j'(x_j)}$.

- The right-hand side does not depend on x_i, the left-hand side does not depend on x_j, so:

$$\frac{r_i'(\lambda \cdot x_i)}{r_i'(x_i)} = c(\lambda) \text{ for some } c(\lambda).$$

13. Let Us Use Homogeneity (cont-d)

- Thus, the derivative $R_i(x_i) \stackrel{\text{def}}{=} r_i'(x_i)$ satisfies the functional equation:

$$R_i(\lambda \cdot x_i) = R_i(x_i) \cdot c(\lambda) \text{ for all } \lambda \text{ and } x_i.$$

- It is known that every continuous solution to this equation has the form $r_i'(x_i) = R_i(x_i) = A_i \cdot x_i^{\alpha_i}$.
- For differentiable functions, this can be proven if we differentiate both sides by λ and take $\lambda = 1$.
- Then, we get $x_i \cdot \dfrac{dR_i}{dc_i} = c \cdot R_i$.
- Separating variables, we get $\dfrac{dR_i}{R_i} = c \cdot \dfrac{dx_i}{x_i}$.
- Integration leads to $\ln(R_i) = c \cdot \ln(x_i) + C_1$ and thus, to the desired formula $r_i'(x_i) = A_i \cdot x_i^{\alpha_i}$.

14. Let Us Use Homogeneity (cont-d)

- Integrating the above expression for $r'_i(x_i)$, we get $r_i(x_i) = a_i \cdot x_i^{\beta_i} + C_i$ and similarly, $r_j(x_j) = a_j \cdot x_j^{\beta_j} + C_j$.
- One can easily check that homogeneity implies that $\beta_i = \beta_j$ and $C_i + C_j = 0$, so

$$r_i(x_i) + r_j(x_j) = a_i \cdot x_i^r + a_j \cdot x_j^r.$$

- By considering a similar substitution between x_i and y, we conclude that $r_{ij}(y) = \text{const} \cdot y^r$.
- So, we indeed get the desired formula

$$r_{ij}(x_i, x_j) = (a_i \cdot x_i^r + a_j \cdot x_j^r)^{1/r}.$$

- By using similar formulas to combine x_{ij} with x_k, etc., we get the desired CES combination function.

15. Possible Application to Copulas

- A 1-D probability distribution of a random variable X can be described by its cdf $F_X(x) \stackrel{\text{def}}{=} \text{Prob}(X \le x)$.
- A 2-D distribution of a random vector (X, Y) can be similarly described by its 2-D cdf

$$F_{XY}(x, y) = \text{Prob}(X \le x \,\&\, Y \le y).$$

- It turns out that we can always describe $F(x, y)$ as

$$F_{XY}(x, y) = C_{XY}(F_X(x), F_Y(y)) \text{ for some } C_{XY}(a, b).$$

- This function C_{XY} is called a *copula*.
- For a joint distribution of several random variables X, Y, ..., Z, we can similarly write

$$F_{XY\ldots Z}(x, y, \ldots, z) \stackrel{\text{def}}{=} \text{Prob}(X \le x \,\&\, Y \le y \,\&\, \ldots \,\&\, Z \le z) =$$

$$C_{XY\ldots Z}(F_X(x), F_Y(y), \ldots, F_Z(z)) \text{ for some } C_{XY\ldots Z}.$$

16. Copulas (cont-d)

- When we have many ($n \gg 1$) random variables, then we need to describe a function of n variables.
- Even if we use two values for each variable, we get 2^n combinations.
- For large n this is astronomically large.
- Thus, a reasonable idea is to approximate the multi-D distribution.
- A reasonable way to approximate is to use 2-D copulas.

17. Copulas (cont-d)

- For example, to describe a joint distribution of three variables X, Y, and Z:
  - we first describe the joint distribution of X and Y as $F_{XY}(x,y) = C_{XY}(F_X(x), F_Y(y))$,
  - and then use a copula $C_{XY,Z}$ to combine it with $F_Z(z)$: $F_{XYZ}(x,y,z) \approx C_{XY,Z}(F_{XY}(x,y), F_Z(z)) =$

$$C_{XY,Z}(C_{XY}(F_X(x), F_Y(y), F_Z(z)).$$

- Such an approximation, when copulas are applied to one another like a vine, are known as *vine copulas*.
- It is reasonable to require that:
  - the result of the vine copula approximation
  - should not depend on the order in which we combine the variables.

18. Copulas (cont-d)

- In particular, for X, Y, Z, and T, we should get the same result in the following two situations:
  - if we first combine X with Y, Z and T, and then combine the two results; or
  - if we first combine X with Z, Y with T, and then combine the two results.
- Thus, we require that for all possible real numbers x, y, z, and t, we get

$$C_{XY,ZT}(C_{XY}(F_X(x), F_Y(y)), C_{ZT}(F_Z(z), F_T(t))) =$$

$$C_{XZ,YT}(C_{XZ}(F_X(x), F_Z(z)), C_{YT}(F_Y(y), F_T(t))).$$

- If we denote $a = F_X(x)$, $b = F_Y(y)$, $c = F_Z(z)$, and $d = F_T(t)$, we conclude that for every a, b, c, and d:

$$C_{XY,ZT}(C_{XY}(a, b), C_{ZT}(c, d)) = C_{XZ,YT}(C_{XZ}(a, c), C_{YT}(b, d)).$$

19. Copulas: Conclusion

- We have:

$$C_{XY,ZT}(C_{XY}(a,b), C_{ZT}(c,d)) = C_{XZ,YT}(C_{XZ}(a,c), C_{YT}(b,d)).$$

- This is exactly the above generalized associativity requirement; thus:
  - if we extend copulas to invertible operations,
  - then we can conclude that copulas can be *re-scaled to associative operations* $\circ$:

$$C(a,b) = f(g(a) \circ h(b)) \text{ for some } f, g, h.$$

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