Towards a General Computation-Oriented Description of Physical Quantities: From Intervals to Graphs to Simplicial Complexes and Their Projective Limits

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1. Main Problem: Introduction

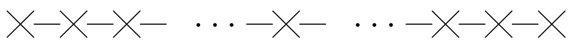
- *One of the main objectives of physics:* predict the future behavior of real-world systems.
- *Fact:* in modern physics, models for space, time, causality, and physical processes in general are very complex.
- *Example:*
  - *physical phenomenon:* a simple space-time;
  - *formalism:* quantum physics;
  - *mathematical description:* a wave function $\psi(M)$ defined on all pseudo-Riemannian manifolds M.
- *Corollary:* prediction-related computations are often extremely time-consuming.
- *Sometimes:* by the time we finish prediction computations, the predicted event has already occurred.
- *Problem:* how can we speed up these computations?

2. An Approach to Solving the Main Problem: Operationalism

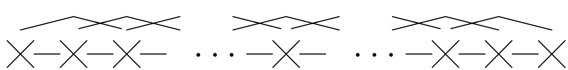
- *Fact:* in modern physics, many quantities used in the corresponding equations are not directly observable.
- *Example:* the wave function $\psi(x)$.
- *Related idea:* restrict ourselves to only computing directly observable quantities.
- *Hope:* by not computing other quantities, we can save computation time.
- *Reason for this hope:* a similar operationalistic approach has been very successful in physics:
  - *special relativity:* started with Einstein's analysis of simultaneity;
  - *general relativity:* Einstein's equivalence principle;
  - *equations of quantum physics:* Heisenberg's matrix equations (motivated by operationalism).

3. Towards Operationalistic Approach to Computational Physics: Binary Domains

- *General idea:* in any real-life measurement, we have a finite set X of possible measurement results.
- *Description of measurement uncertainty:* $a \sim b \leftrightarrow$ the same object can lead to both a and b.
- *Physical example:* temperature t°; values 0, 1, ..., 100; *measurement accuracy:* $\pm 0.5^\circ$.
- $X = \{0, 1, 2, 3, \ldots, 100\}$; $a \sim b \leftrightarrow |a - b| \le 1$.



- *Modified example:* measurement accuracy $\pm 1^\circ$.
- $X = \{0, 1, 2, 3, \ldots, 100\}$; $a \sim b \leftrightarrow |a - b| \le 2$.



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4. Towards Operationalistic Approach to Computational Physics: Binary Domains (continued)

- *Counting* up to n: $X = \{1, 2, \ldots, n-1, \text{many}\}$:

$$0 \quad 1 \quad \cdots \quad k \quad \cdots \quad (n-1)\ \text{many}$$

- *Binary questions:* "yes" (1), "no" (0), "unknown" (U); $X = \{0, 1, U\}$, $0 \sim U \sim 1$.

- Repeated "yes"-"no" measurements: 5 possible outcomes: 0_1, 1_1, $U_1 0_2$, $U_1 1_2$, and $U_1 U_2$.
  - If the actual value is 0, we can get 0_1, $U_1 0_2$, $U_1 U_2$;
  - if the actual value is 1, we can get 1_1, $U_1 1_2$, $U_1 U_2$.

- *General case:* graph (*web*) $\langle X, \sim \rangle$.

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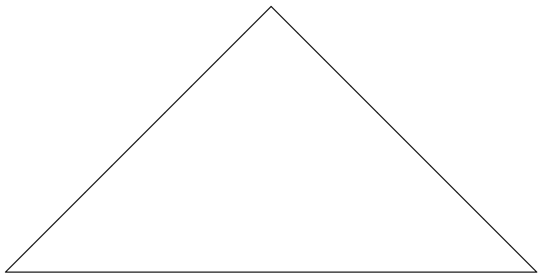
5. A More Adequate Description: Simplicial Complexes

- *Previously:* we only considered compatibility of pairs of measurement results.
- *Natural idea:* consider compatibility of triples, etc.
- *Formalization:*
  - a set $S \subseteq X$ is *compatible*
  - if for some object, all values from S are possible after measurement.
- *Simplicial complex:* a pair $\langle X, \mathcal{S} \rangle$, where $X \subseteq \mathcal{S} \subseteq 2^X$ is the class of all compatible sets.

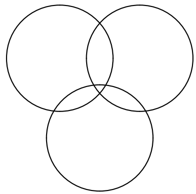
6. Simplicial Complex: Example 1

- *Example 1:* $X_i \cap X_j \neq \emptyset$ but $X_1 \cap X_2 \cap X_3 = \emptyset$.
- *Corresponding simplicial complex:* empty triangle
  - $X = \{x_1, x_2, x_3\}$,
  - $\mathcal{S} = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3\}\}$.
- *Illustration:*



7. Simplicial Complex: Example 2

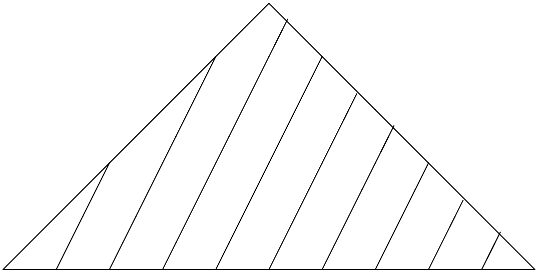
- *Example 2:* $X_1 \cap X_2 \cap X_3 \neq \emptyset$.



- *Corresponding simplicial complex:* filled triangle

$$X = \{x_1, x_2, x_3\},$$

$$\mathcal{S} = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3\}, \{x_1, x_2, x_3\}\}.$$



8. How to Describe Actual Values of Measured Quantities

- *Objective:* describe actual values.
- *Problem:* single measurement leads to approximate value.
- *Solution:* consider a sequence of more and more accurate measuring instruments.
- *Relation:* let X describes results of first k measurements and X' results of $l > k$ measurements.
- The forgetful functor $\pi_{lk} : X' \to X$ is a *projection:*
  - if $a' \sim' b'$, then $\pi(a') \sim \pi(b')$;
  - if $a \sim b$, then $\exists a', b'$ s.t. $\pi(a') = a$, $\pi(b') = b$, and $a' \sim' b'$.
- *Definition:* $X_1 \stackrel{\pi_{2,1}}{\longleftarrow} X_2 \stackrel{\pi_{3,2}}{\longleftarrow} X_3 \stackrel{\pi_{4,3}}{\longleftarrow} \ldots$
- *Actual values:* $x = (x_1, x_2, \ldots)$ s.t. $\pi_{21}(x_2) = x_1, \ldots$

9. Actual Values: Properties and Examples

- *Equivalence:* $a \sim b$ iff $a_i \sim_i b_i$ for all i.
- *Neighborhoods:* $N_n(a) = \{b \,|\, b \sim_n a\}$.
- *Limit:* $a^{(k)} \to a$ iff $\forall n\, \exists m\, \forall k > m\, (a_n^{(k)} \sim_n a)$.
- *Real numbers:* naturally come from intervals.
- *Actually:* we also get $-\infty$ and $+\infty$.
- R^n*:* naturally comes from n-dimensional boxes.
- *"yes"-"no" questions:*
  - X_1: $0 \sim U \sim 1$;
  - X_2: $0 \sim U0 \sim UU \sim U1 \sim 1$, $0 \sim UU \sim 1$;
  - ...
  - projective limit: $0 \sim U \sim 1$.

10. Unusual Property: Compactness

- *General property:* every sequence has a convergent subsequence.
- *Example:* instead of R, we have a compactification $R \cup \{-\infty, +\infty\}$.
- *Potential application:* inverse problems.
- *Description:* we observe $f(x)$ for some continuous $f : X \to Y$; we want to reconstruct x.
- *Example:* signal from its distortion.
- *Problem:* even if f is 1-1, f^{-1} is discontinuous, so close y lead to different x.
- *Solution:* for compact X, f^{-1} is continuous.
- *Similar property:* $\sim$ is transitive iff

$$\forall n\, \exists m\, ((a \sim_m b \,\&\, b \sim_m c) \to (a \sim_n b).$$

11. Functions

- *Meaning:* $f : A \to B$ means that:
  - once we know an approximation a_n to a,
  - we can find some approximation b_m to b.
- *Definition:* a function $f : A \to B$ is a mapping from $\cup A_n$ to $\cup B_n$ s.t.:
  - $a \sim a'$ implies $f(a) \sim f(a')$;
  - if $a = \pi(a')$, then $f(a) = \pi(f(a'))$.
- *Comment:* functions may be partial, so the results do not converge.
- *Everywhere defined:* if $f : X \to R$ is everywhere defined, then f is continuous:

$$\forall n\, \exists m\, ((x_m \sim_m x'_m) \to f(x_m) \sim_n f(x'_m)).$$

12. Summary

- *Idea:* restrict ourselves to directly observable results.
- *Measuring instrument:* a finite graph in which:
  - vertices are possible measurement results and
  - vertices a and b are connected by an edge iff a and b can come from measuring the same quantity.
- *Physical quantity:* a sequence of more and more accurate measuring instruments.
- *Resulting mathematical representation:* a projective limits of the corresponding graphs (or complexes).
- *Computational advantage:*
  - *mathematical fact:* higher order objects (functions, operators, etc.) described by similar graphs;
  - *computational advantage:* such higher order objects are algorithmically computable.

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13. Acknowledgments

This work was supported in part:

- by NSF grant HRD-0734825,
- by Texas Department of Transportation Research Project No. 0-5453,
- by the Japan Advanced Institute of Science and Technology (JAIST) International Joint Research Grant 2006-08, and
- by the Max Planck Institut für Mathematik.

14. Further Reading

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