Metrization Theorem for Space-Times: A Constructive Solution to Urysohn's Problem

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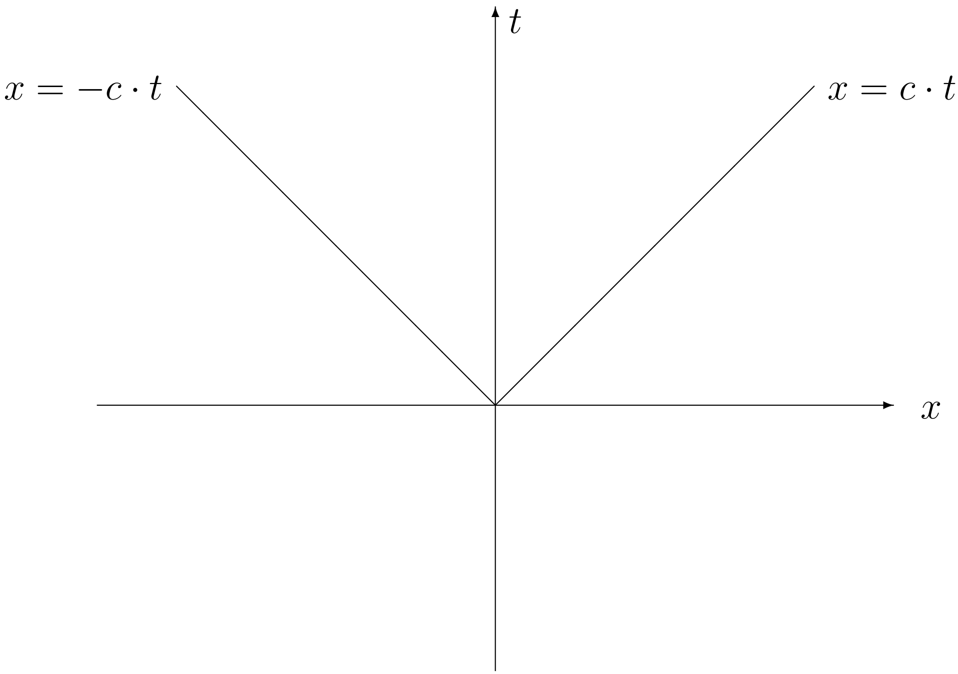
1. Urysohn's Lemma and Urysohn's Metrization Theorem: Reminder

- *Who, when:* early 1920s, Pavel Urysohn.
- *Claim for fame:* Urysohn's Lemma is "first non-trivial result of point set topology".
- *Condition:* X is a normal topological space X, A and B are disjoint closed sets.
- *Conclusion:* there exists $f : X \to [0,1]$ s.t. $f(A) = \{0\}$ and $f(B) = \{1\}$.
- *Reminder:* normal means that every two disjoint closed sets have disjoint open neighborhoods.
- *Application:* every normal space with countable base is metrizable.
- *Comment:* actually, every regular Hausdorff space with countable base is metrizable.

2. Extension to Space-Times: Urysohn's Problem

- *Fact:* a few years before that, in 1919, Einstein's GRT has been experimentally confirmed.
- *Corresponding structure:* topological space with an order (casuality).
- *Urysohn's problem:* extend his lemma and metrization theorem to (causality-)ordered topological spaces.
- *Tragic turn of events:* Urysohn died in 1924.
- *Follow up:* Urysohn's student Vadim Efremovich; Efremovich's student Revolt Pimenov; Pimenov's students.
- *Other researchers:* H. Busemann (US), E. Kronheimer and R. Penrose (UK).
- *Result:* by the 1970s, space-time versions of Uryson's lemma and metrization theorem have been proven.

3. Causality: A Reminder



4. Urysohn's Problem: Remaining Issues

- *Main issue:* the 1970s results are not constructive.
- *Why this is important:* we want useful applications to physics.
- *What we have now:* theoretical existence of a pseudo-metric.
- *What we need:* an algorithm generating such a metric based on the empirical causality.
- *Also:* we need a physically relevant constructive description of a causality-type ordering relation.
- *Our objective:*
  - to propose such a description, and
  - to prove constructive space-time versions of the Uryson's lemma and metrization theorem.

5. Space-Time Models: Reminder

- *Theoretical relation:* (transitive) causality $a \preceq b$.
- *Problem:* events are not located exactly: $\widetilde{a} \approx a$, $\widetilde{b} \approx b$.
- *Practical relation:* kinematic casuality $a \prec b$.
- *Meaning:* every event in some small neighborhood of b causally follows a, i.e., $b \in \text{Int}(a^+)$.
- *Properties of $\prec$:* $\prec$ is transitive; $a \not\prec a$;

$$\forall a\, \exists \underline{a}, \overline{a}\, (\underline{a} \prec a \prec \overline{a}); \quad a \prec b \Rightarrow \exists c\, (a \prec c \prec b);$$

$$a \prec b, c \Rightarrow \exists d\, (a \prec d \prec b, c); \quad b, c \prec a \Rightarrow \exists d\, (b, c \prec d \prec a).$$

- *Alexandrov topology:* with intervals as the base:

$$(a, b) \stackrel{\text{def}}{=} \{c : a \prec c \prec b\}.$$

- *Description of causality:* $a \preceq b \stackrel{\text{def}}{\equiv} b \in \overline{a^+}$.
- *Additional property:* $b \in \overline{a^+} \Leftrightarrow a \in \overline{b^-}$.

6. Space-Time Analog of a Metric

- *Traditional metric:* a function $\rho : X \times X \to R_0^+$ s.t.

$$\rho(a,b) = 0 \Leftrightarrow a = b;$$

$$\rho(a,b) = \rho(b,a);$$

$$\rho(a,c) \le \rho(a,b) + \rho(b,c).$$

- *Physical meaning:* the length of the shortest path between a and b.

- *Kinematic metric:* a function $\tau : X \times X \to R_0^+$ s.t.

$$\tau(a,b) > 0 \Leftrightarrow a \prec b;$$

$$a \prec b \prec c \Rightarrow \tau(a,c) \ge \tau(a,b) + \tau(b,c).$$

- *Physical meaning:* the longest (= proper) time from event a to event b.

- *Explanation:* when we speed up, time slows down.

7. Space-Time Analogs of Urysohn's Lemma and Metrization Theorem

- *Main condition:* the kinematic space is *separable*, i.e., there exists a countable dense set $\{x_1, x_2, \ldots, x_n, \ldots\}$.
- *Condition of the lemma:* X is separable, and $a \prec b$.
- *Lemma:* $\exists$ a cont. $\preceq$-increasing f-n $f_{(a,b)} : X \to [0,1]$ s.t. $f_{(a,b)}(x) = 0$ for $a \not\prec x$ and $f_{(a,b)}(x) = 1$ for $b \preceq x$.
- *Relation to the original Urysohn's lemma:* $f_{(a,b)}$ separates disjoint closed sets $-a^+$ and $\overline{b^+}$.
- *Condition of the theorem:* $(X, \prec)$ is a separable kinematic space.
- *Theorem:* there exists a continuous metric τ which generates the corresponding relation $\prec$.
- *Corollary:* τ also generates the corresponding topology.

8. How the (Non-Constructive) Space-Time Metrization Theorem Is Proved

- *First lemma:* for every x, there exists a $\prec$-monotonic function $f_x : X \to [0,1]$ for which $f_x(b) > 0 \Leftrightarrow x \prec b$.
- *Proof:* $\exists y_i \searrow x$; take $f_x(b) = \sum\limits_{i=1}^{\infty} 2^{-i} \cdot f_{(x,y_i)}(b)$.
- *Second lemma:* for every x, there exists a $\prec$-monotonic function $g_x : X \to [0,1]$ for which $g_x(a) > 0 \Leftrightarrow a \prec x$.
- *Proof:* similar.
- *Resulting metric:* for a countable everywhere dense sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$, take

$$\tau(a,b) = \sum_{i=1}^{\infty} 2^{-i} \cdot \min(g_{x_i}(a), f_{x_i}(b)).$$

9. Constructive Causality: What Does It mean?

- *How to find causality:* we send a signal at event a:
  - if this signal is detected at b, then $a \preceq b$;
  - if this signal is not detected at b, then $a \not\preceq b$.
- *Practical problem:* we can only locate an event with a certain accuracy.
- *Result:* we have 3 options:
  - if the signal is detected in the entire vicinity of b, then $a \prec b$;
  - if no signal is detected in the entire vicinity of b, then $a \not\preceq b$;
  - in all other cases, we do not know.
- *Conclusion:* we have relations $\prec_n$ corr. to increasing location accuracy, so $a \prec b \Leftrightarrow \exists n\,(a \prec_n b)$.

10. Constructive Causality: Towards a Precise Definition

- $\prec$ is transitive; $a \not\prec a$;

$$\forall a\, \exists \underline{a}, \overline{a}\, (\underline{a} \prec a \prec \overline{a}); \quad a \prec b \Rightarrow \exists c\, (a \prec c \prec b);$$

$$a \prec b, c \Rightarrow \exists d\, (a \prec d \prec b, c); \quad b, c \prec a \Rightarrow \exists d\, (b, c \prec d \prec a).$$

- *Main difference:* $\exists$ is understood constructively.
- If $a \prec b$, then $\forall c\, (a \prec c \vee b \not\preceq c)$.
- There exists a sequence $\{x_i\}$ for which

$$a \prec b \Rightarrow \exists i\, (a \prec x_i \prec b).$$

- There exists a decidable ternary relation $x_i \prec_n x_j$ for which

$$x_i \prec x_j \Leftrightarrow \exists n\, (x_i \prec_n x_j).$$

- *Comment:* decidable means that $x_i \prec_n x_j \vee x_i \not\prec_n x_j$.

11. Constructive Space-Time Version of Urysohn's Lemma: Proof

- *Objective:* given $a \prec b$, design a monotonic function $f_{(a,b)} : X \to [0,1]$ s.t. $f_{(a,b)}(-a^+) = 0$ and $f_{(a,b)}(b^+) = 1$.
- *Auxiliary result:* $a \prec b \Rightarrow \exists c\,(a \prec c \prec b) \Rightarrow \exists i\,(a \prec x_i \prec c \prec b) \Rightarrow \exists i\,(a \prec x_i \prec b)$.
- *Part 1:* define $\prec$-monotonic values $\gamma(p/2^q)$, $p \le 2^q$.
- $q = 0$: $\gamma(0) = a$ and $\gamma(1) = b$.
- *From q to $q+1$:* take x_i s.t. $\gamma(p/2^q) \prec x_i \prec \gamma((p+1)/2^q)$ as midpoint value $\gamma((p+1/2)/2^q) \equiv \gamma((2p+1)/2^{q+1})$.
- *Part 2:* compute $f_{(a,b)}(x) \stackrel{\text{def}}{=} \sup\{r : \gamma(r) \prec x\}$.
- *Idea:* $\gamma(p/2^q) \prec x \vee \gamma((p+1)/2^q) \not\preceq x$, hence
$$f_{(a,b)}(x) > p/2^q \vee f_{(a,b)}(x) \le (p+1)/2^q.$$
- *Algorithm:* so, we compute $f_{(a,b)}(x)$ with accuracy 2^{-q}.

12. Constructive Space-Time Metrization Theorem: Proof

- *Reminder:* for all $a \prec b$, we have a monotonic function $f_{(a,b)} : X \to [0,1]$ s.t. $f_{(a,b)}(-a^+) = 0$ and $f_{(a,b)}(b^+) = 1$.
- *Reminder:* relation $x_i \prec_n x_j$ is decidable.
- *Step 1:* for every i, we define $f_{x_i} : X \to [0,1]$ as follows:

$$f_{x_i}(b) \stackrel{\text{def}}{=} \sum_{j,n:\ x_i \prec_n x_j} 2^{-j} \cdot 2^{-n} \cdot f_{(x_i,x_j)}(b).$$

- *Easy to prove:* $f_{x_i}(b)$ is $\preceq$-monotonic and

$$f_{x_i}(b) > 0 \Leftrightarrow x_i \prec b.$$

- Similarly, we define functions $g_{x_i}(a)$.
- *Resulting kinematic metric:* same as before:

$$\tau(a,b) = \sum_{i=1}^{\infty} 2^{-i} \cdot \min(g_{x_i}(a), f_{x_i}(b)).$$

13. Auxiliary Results

- *Time coordinate:*

$$t(b) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} 2^{-i} \cdot f_{x_i}(b).$$

- *Comment:* since $f_{x_i}(b) \in [0,1]$, this is constructively defined.

- *Properties:*
  - $a \prec b \Rightarrow t(a) < t(b)$;
  - $a \preceq b \Rightarrow t(a) \le t(b)$.

- *Standard metric:*

$$\rho(a,b) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} 2^{-i} \cdot |f_{x_i}(a) - f_{x_i}(b)|.$$

14. Symmetries: A Remaining Challenge

- *So far:* given space-time X, we designed a metric τ.
- *Symmetry:* one of the most important notions of physics.
- *Situation:* space-time has symmetries.
- *Find:* τ which is invariant w.r.t. these symmetries.
- *Simple case:* finite symmetry group G.
- *Solution:* $\tau_{\text{inv}}(a,b) \stackrel{\text{def}}{=} \sum\limits_{g \in G} \tau(g(a), g(b))$.
- *Important case:* X is an ordered group and a kinematic space, with compact intervals.
- *Known:* there exists a left-invariant metric $\tau(a,b)$.
- *Proof:* $\tau(a,b) = \mu_H(\{c : a \preceq c \preceq b\})$ where μ_H is the (left-invariant) Haar measure.
- *Open problem:* constructivize such results; maybe R. Mines' and F. Richman's ideas can help?

15. Acknowledgments

This work was supported in part:

- by NSF grant HRD-0734825,
- by Texas Department of Transportation Research Project No. 0-5453,
- by the Japan Advanced Institute of Science and Technology (JAIST) International Joint Research Grant 2006-08, and
- by the Max Planck Institut für Mathematik.