Selecting the Best Location for a Meteorological Tower: A Case Study of Multi-Objective Constraint Optimization

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1. Outline

- *Case study:* meteorological tower.
- *This case* is an example of multi-criteria optimization, when we need to maximize several objectives $x_1, \ldots, x_n$.
- *Traditional approach* to multi-objective optimization: maximize a weighted combination $\sum\limits_{i=1}^{n} w_i \cdot x_i$.
- *Specifics of our case:* constraints $x_i > x_i^{(0)}$ or $x_i < x_i^{(0)}$.
- *Equiv.:* $y_i > 0$, where $y_i \stackrel{\text{def}}{=} x_i - x_i^{(0)}$ or $y_i = x_i^{(0)} - x_i$.
- *Limitations* of using the traditional approach under constraints.
- *Scale invariance:* a brief description.
- *Main result:* scale invariance leads to a new approach: maximize $\sum\limits_{i=1}^{n} w_i \cdot \ln(y_i) = \sum\limits_{i=1}^{n} w_i \cdot \ln \left| x_i - x_i^{(0)} \right|$.

2. Case Study

- *Objective:* select the best location of a sophisticated multi-sensor meteorological tower.
- *Constraints:* we have several criteria to satisfy.
- *Example:* the station should not be located too close to a road.
- *Motivation:* the gas flux generated by the cars do not influence our measurements of atmospheric fluxes.
- *Formalization:* the distance x_1 to the road should be larger than a threshold t_1: $x_1 > t_1$, or $y_1 \stackrel{\text{def}}{=} x_1 - t_1 > 0$.
- *Example:* the inclination x_2 at the tower's location should be smaller than a threshold t_2: $x_2 < t_2$.
- *Motivation:* otherwise, the flux determined by this inclination and not by atmospheric processes.

3. General Case

- *In general:* we have several differences $y_1, \ldots, y_n$ all of which have to be non-negative.
- For each of the differences y_i, the larger its value, the better.
- Our problem is a typical setting for *multi-criteria optimization.*
- A most widely used approach to multi-criteria optimization is *weighted average*, where
  - we assign weights $w_1, \ldots, w_n > 0$ to different criteria y_i and
  - select an alternative for which the weighted average

$$w_1 \cdot y_1 + \ldots + w_n \cdot y_n$$

    attains the largest possible value.

4. Limitations of the Weighted Average Approach

- *In general:* the weighted average approach often leads to reasonable solutions of the multi-criteria problem.
- *In our problem:* we have an additional requirement – that all the values y_i must be positive. So:
  - when selecting an alternative with the largest possible value of the weighted average,
  - we must only compare solutions with $y_i > 0$.
- *We will show:* under the requirement $y_i > 0$, the weighted average approach is not fully satisfactory.
- *Conclusion:* we need to find a more adequate solution.

5. Limitations of the Weighted Average Approach: Details

- The values y_i come from measurements, and measurements are never absolutely accurate.
- The results $\widetilde{y}_i$ of the measurements are not exactly equal to the actual (unknown) values y_i.
- *If:* for some alternative $y = (y_1, \ldots, y_n)$
  - we measure the values y_i with higher and higher accuracy and,
  - based on the measurement results $\widetilde{y}_i$, we conclude that y is better than some other alternative y'.
- *Then:* we expect that the actual alternative y is indeed better than y' (or at least of the same quality).
- Otherwise, we will not be able to make any meaningful conclusions based on real-life measurements.

6. The Above Natural Requirement Is Not Always Satisfied for Weighted Average

- *Simplest case:* two criteria y_1 and y_2, w/weights $w_i > 0$.
- If $y_1, y_2, y'_1, y'_2 > 0$, and $w_1 \cdot y_1 + w_2 \cdot y_2 > w_1 \cdot y'_1 + w_2 \cdot y'_2$, then $y = (y_1, y_2) \succ y' = (y'_1, y'_2)$.
- If $y_1 > 0$, $y_2 > 0$, and at least one of the values y'_1 and y'_2 is non-positive, then $y = (y_1, y_2) \succ y' = (y'_1, y'_2)$.
- Let us consider, for every $\varepsilon > 0$, the tuple $y(\varepsilon) \stackrel{\text{def}}{=} (\varepsilon, 1 + w_1/w_2)$, and $y' = (1, 1)$.
- In this case, for every $\varepsilon > 0$, we have
$$w_1 \cdot y_1(\varepsilon) + w_2 \cdot y_2(\varepsilon) = w_1 \cdot \varepsilon + w_2 + w_2 \cdot \frac{w_1}{w_2} = w_1 \cdot (1+\varepsilon) + w_2$$
and $w_1 \cdot y'_1 + w_2 \cdot y'_2 = w_1 + w_2$, hence $y(\varepsilon) \succ y'$.
- However, in the limit $\varepsilon \to 0$, we have $y(0) = \left(0, 1 + \dfrac{w_1}{w_2}\right)$, with $y(0)_1 = 0$ and thus, $y(0) \prec y'$.

7. Towards a Precise Description

- Each alternative is characterized by a tuple of n positive values $y = (y_1, \ldots, y_n)$.
- Thus, the set of all alternatives is the set $(R^+)^n$ of all the tuples of positive numbers.
- For each two alternatives y and y', we want to tell whether
  - y is better than y' (we will denote it by $y \succ y'$ or $y' \prec y$),
  - or y' is better than y ($y' \succ y$),
  - or y and y' are equally good ($y' \sim y$).
- *Natural requirement:* if y is better than y' and y' is better than y'', then y is better than y''.
- The relation $\succ$ must be transitive.

8. Towards a Precise Description (cont-d)

- *Reminder:* the relation $\succ$ must be transitive.
- Similarly, the relation $\sim$ must be transitive, symmetric, and reflexive ($y \sim y$), i.e., be an *equivalence relation.*
- *An alternative description:* a transitive *pre-ordering* relation $a \succeq b \Leftrightarrow (a \succ b \vee a \sim b)$ s.t. $a \succeq b \vee b \succeq a$.
- Then, $a \sim b \Leftrightarrow (a \succeq b) \,\&\, (b \succeq a)$, and

$$a \succ b \Leftrightarrow (a \succeq b) \,\&\, (b \not\succeq a).$$

- *Additional requirement:*
  - *if* each criterion is better,
  - *then* the alternative is better as well.
- *Formalization:* if $y_i > y'_i$ for all i, then $y \succ y'$.

9. Scale Invariance: Motivation

- *Fact:* quantities y_i describe completely different physical notions, measured in completely different units.
- *Examples:* wind velocities measured in m/s, km/h, mi/h; elevations in m, km, ft.
- Each of these quantities can be described in many different units.
- A priori, we do not know which units match each other.
- Units used for measuring different quantities may not be exactly matched.
- It is reasonable to require that:
  - if we simply change the units in which we measure each of the corresponding n quantities,
  - the relations $\succ$ and $\sim$ between the alternatives $y = (y_1, \ldots, y_n)$ and $y' = (y'_1, \ldots, y'_n)$ do not change.

10. Scale Invariance: Towards a Precise Description

- *Situation:* we replace:
  - a unit in which we measure a certain quantity q
  - by a new measuring unit which is $\lambda > 0$ times smaller.
- *Result:* the numerical values of this quantity increase by a factor of λ: $q \to \lambda \cdot q$.
- *Example:* 1 cm is $\lambda = 100$ times smaller than 1 m, so the length $q = 2$ becomes $\lambda \cdot q = 2 \cdot 100 = 200$ cm.
- Then, scale-invariance means that for all $y, y' \in (R^+)^n$ and for all $\lambda_i > 0$, we have
  - $y = (y_1, \ldots, y_n) \succ y' = (y'_1, \ldots, y'_n)$ implies $(\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n) \succ (\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n)$,
  - $y = (y_1, \ldots, y_n) \sim y' = (y'_1, \ldots, y'_n)$ implies $(\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n) \sim (\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n)$.

11. Formal Description

- By a *total pre-ordering relation* on a set Y, we mean
  - a pair of a transitive relation $\succ$ and an equivalence relation $\sim$ for which,
  - for every $y, y' \in Y$, exactly one of the following relations hold: $y \succ y'$, $y' \succ y$, or $y \sim y'$.
- We say that a total pre-ordering is *non-trivial* if there exist y and y' for which $y \succ y'$.
- We say that a total pre-ordering relation on $(R^+)^n$ is:
  - *monotonic* if $y'_i > y_i$ for all i implies $y' \succ y$;
  - *continuous* if
    - whenever we have a sequence $y^{(k)}$ of tuples for which $y^{(k)} \succeq y'$ for some tuple y', and
    - the sequence $y^{(k)}$ tends to a limit y,
    - then $y \succeq y'$.

12. Main Result

Theorem. *Every non-trivial monotonic scale-inv. continuous total pre-ordering relation on* $(R^+)^n$ *has the form:*

$$y' = (y'_1, \ldots, y'_n) \succ y = (y_1, \ldots, y_n) \Leftrightarrow \prod_{i=1}^{n} (y'_i)^{\alpha_i} > \prod_{i=1}^{n} y_i^{\alpha_i};$$

$$y' = (y'_1, \ldots, y'_n) \sim y = (y_1, \ldots, y_n) \Leftrightarrow \prod_{i=1}^{n} (y'_i)^{\alpha_i} = \prod_{i=1}^{n} y_i^{\alpha_i},$$

for some constants $\alpha_i > 0$.

Comment: Vice versa,

- for each set of values $\alpha_1 > 0, \ldots, \alpha_n > 0$,
- the above formulas define a monotonic scale-invariant continuous pre-ordering relation on $(R^+)^n$.

13. Practical Conclusion

- *Situation:*
  - we need to select an alternative;
  - each alternative is characterized by characteristics $y_1, \ldots, y_n$.
- *Traditional approach:*
  - we assign the weights w_i to different characteristics;
  - we select the alternative with the largest value of $\sum\limits_{i=1}^{n} w_i \cdot y_i$.
- *New result:* it is better to select an alternative with the largest value of $\prod\limits_{i=1}^{n} y_i^{w_i}$.
- *Equivalent reformulation:* select an alternative with the largest value of $\sum\limits_{i=1}^{n} w_i \cdot \ln(y_i)$.

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15. Proof: Part 1

- Due to scale-invariance, for every $y_1, \ldots, y_n, y'_1, \ldots, y'_n$, we can take $\lambda_i = \dfrac{1}{y_i}$ and conclude that

$$(y'_1, \ldots, y'_n) \sim (y_1, \ldots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \ldots, \frac{y'_n}{y_n}\right) \sim (1, \ldots, 1).$$

- Thus, to describe the equivalence relation $\sim$, it is sufficient to describe $\{z = (z_1, \ldots, z_n) : z \sim (1, \ldots, 1)\}$.

- Similarly,

$$(y'_1, \ldots, y'_n) \succ (y_1, \ldots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \ldots, \frac{y'_n}{y_n}\right) \succ (1, \ldots, 1).$$

- Thus, to describe the ordering relation $\succ$, it is sufficient to describe the set $\{z = (z_1, \ldots, z_n) : z \succ (1, \ldots, 1)\}$.

- Similarly, it is also sufficient to describe the set

$$\{z = (z_1, \ldots, z_n) : (1, \ldots, 1) \succ z\}.$$

16. Proof: Part 2

- *To simplify:* take logarithms $Y_i = \ln(y_i)$, and sets

$$S_\sim = \{Z : z = (\exp(Z_1), \ldots, \exp(Z_n)) \sim (1, \ldots, 1)\},$$

$$S_\succ = \{Z : z = (\exp(Z_1), \ldots, \exp(Z_n)) \succ (1, \ldots, 1)\};$$

$$S_\prec = \{Z : (1, \ldots, 1) \succ z = (\exp(Z_1), \ldots, \exp(Z_n))\}.$$

- Since the pre-ordering relation is total, for Z, either $Z \in S_\sim$ or $Z \in S_\succ$ or $Z \in S_\prec$.
- *Lemma:* $S_\sim$ is closed under addition:
  - $Z \in S_\sim$ means $(\exp(Z_1), \ldots, \exp(Z_n)) \sim (1, \ldots, 1)$;
  - due to scale-invariance, we have

$$(\exp(Z_1 + Z_1'), \ldots) = (\exp(Z_1) \cdot \exp(Z_1'), \ldots) \sim (\exp(Z_1'), \ldots);$$

  - also, $Z' \in S_\sim$ means $(\exp(Z_1'), \ldots) \sim (1, \ldots, 1)$;
  - since $\sim$ is transitive,

$$(\exp(Z_1 + Z_1'), \ldots) \sim (1, \ldots) \text{ so } Z + Z' \in S_\sim.$$

17. Proof: Part 3

- *Reminder:* the set $S_\sim$ is closed under addition;
- Similarly, $S_\prec$ and $S_\succ$ are closed under addition.
- *Conclusion:* for every integer $q > 0$:
  - if $Z \in S_\sim$, then $q \cdot Z \in S_\sim$;
  - if $Z \in S_\succ$, then $q \cdot Z \in S_\succ$;
  - if $Z \in S_\prec$, then $q \cdot Z \in S_\prec$.
- Thus, if $Z \in S_\sim$ and $q \in N$, then $(1/q) \cdot Z \in S_\sim$.
- We can also prove that $S_\sim$ is closed under $Z \to -Z$:
  - $Z = (Z_1, \ldots) \in S_\sim$ means $(\exp(Z_1), \ldots) \sim (1, \ldots)$;
  - by scale invariance, $(1, \ldots) \sim (\exp(-Z_1), \ldots)$, i.e., $-Z \in S_\sim$.
- Similarly, $Z \in S_\succ \Leftrightarrow -Z \in S_\prec$.
- So $Z \in S_\sim \Rightarrow (p/q) \cdot Z \in S_\sim$; in the limit, $x \cdot Z \in S_\sim$.

18. Proof: Final Part

- *Reminder:* $S_\sim$ is closed under addition and multiplication by a scalar, so it is a linear space.
- *Fact:* $S_\sim$ cannot have full dimension n, since then all alternatives will be equivalent to each other.
- *Fact:* $S_\sim$ cannot have dimension $< n-1$, since then:
  - we can select an arbitrary $Z \in S_\prec$;
  - connect it w/$-Z \in S_\succ$ by a path γ that avoids $S_\sim$;
  - due to closeness, $\exists \gamma(t^*)$ in the limit of $S_\succ$ and $S_\prec$;
  - thus, $\gamma(t^*) \in S_\sim$ – a contradiction.
- Every $(n-1)$-dim lin. space has the form $\sum\limits_{i=1}^{n} \alpha_i \cdot Y_i = 0$.
- Thus, $Y \in S_\succ \Leftrightarrow \sum \alpha_i \cdot Y_i > 0$, and

$$y \succ y' \Leftrightarrow \sum \alpha_i \cdot \ln(y_i/y'_i) > 0 \Leftrightarrow \prod y_i^{\alpha_i} > \prod {y'_i}^{\alpha_i}.$$