Towards a Physically Meaningful Definition of Computable Discontinuous and Multi-Valued Functions (Constraints)

Martine Ceberio, Olga Kosheleva, and Vladik Kreinovich

University of Texas at El Paso
El Paso, Texas 79968, USA
mceberio@utep.edu, olgak@utep.edu
vladik@utep.edu

1. Need to Define Computable Discontinuous Functions

- Many physical phenomena include discontinuous dependencies $y = f(x)$ (“jumps”).
- *Examples:* phase transitions, quantum transitions.
- In other physical situations, for some values x, we may have several possible values y.
- From the mathematical viewpoint, this means that the relation between x and y is no longer a function.
- It is a *relation*, aka *constraint* $R \subseteq X \times Y$, or a *multi-valued function*.
- We thus need to know when a discontinuous and/or multi-valued function to be computable.
- Alas, the current definitions of computable functions are mostly limited to continuous case.

2. Computable Numbers and Metric Spaces: Reminder

- Intuitively, a real number x is *computable* if we can compute it with any desired accuracy.
- Formally, x is *computable* if $\exists$ an algorithm that, given $n \in \mathbb{N}$, returns a rational number r_n s.t. $|x - r_n| \le 2^{-n}$.
- A similar notion of computable elements can be defined for general metric spaces.
- At each moment of time, we only have a finite amount of information about x.
- Based on this information, we produce an approximation corresponding to this information.
- Any information can be represented, in the computer, as a sequence of 0s and 1s.

3. Computable Metric Spaces (cont-d)

- Any 0-1 sequence can be, in turn, interpreted as a binary integer n.
- Let $\widetilde{x}_n$ denote an approximation corresponding to an integer n.
- So, we require that in a computable metric space, there is a sequence of such approximating elements $\{\widetilde{x}_n\}$.
- Computable means, in particular, that the distance $d_X(\widetilde{x}_n, \widetilde{x}_m)$ between such elements should be computable.
- A metric space X with a sequence $\{\widetilde{x}_n\}$ is called *computable* if $\exists$ an algorithm $m, n \to d_X(\widetilde{x}_m, \widetilde{x}_n)$.
- An element $x \in X$ is called *computable* if there exists an algorithm $n \to k_n$ s.t. $d_X(\widetilde{x}_{k_n}, x) \le 2^{-n}$.

4. Computable Functions

- A f-n $f : X \to Y$ from comp. metric space X to comp. metric space Y is *computable* if $\exists$ algorithm s.t.:
  - it uses x as an input, and
  - it computes, for each integer n, a 2^{-n}-approximation y_k to $f(x)$.
- By "uses x as an input", we mean that this algorithm can request, for each m, a 2^{-m}-approximation x_ℓ to x.
- Alas, all the functions computable according to this definition are continuous.
- Thus, we cannot use this definition to check how well we can compute a discontinuous function.

5. Continuity Explained

- Continuity of continuous functions is easy to understand.
- Lets us consider a simple discontinuous function $f(x) = \text{sign}(x)$:
  - $\text{sign}(x) = 1$ for $x > 0$;
  - $\text{sign}(x) = 0$ for $x = 0$;
  - $\text{sign}(x) = -1$ for $x < 0$.
- Let us assume that we can compute $\text{sign}(x)$ with accuracy 2^{-2}.
- Then we would be able, given a comp. real number x, to tell whether $x = 0$.
- This is known to be algorithmically impossible.

6. Computable Compact Set

- In analyzing computability, it is often useful to start with *pre-compact* metric spaces X, where:
  - for every positive real number $\varepsilon > 0$,
  - there exists a finite ε-*net* L, i.e.,

$$\forall x \in X \, \exists y \in L \, (d_X(x, y) \leq \varepsilon).$$

- A pre-compact set is *compact* if every converging sequence has a limit.
- A compact metric space X *computable compact* if:
  - X is a computable metric space, and
  - there exists an algorithm that, given an integer n, returns a 2^{-n}-net L_n for X.

7. Simplifying Comment

- Functions can also be undefined for some inputs x.
- This is easy to repair: if a relation is not everywhere defined:
  - we can make it everywhere defined
  - if we consider, instead of the original set X, a projection of R on this set.
- For example, a function $\sqrt{x}$:
  - is not everywhere defined on the real line, but
  - it is everywhere defined on the set of all non-negative real numbers.
- Thus, without losing generality, we can assume that our relation R is everywhere defined:

$$\forall x \in X \, \exists y \in Y \, ((x, y) \in R).$$

8. Analysis of the Problem

- From the physical viewpoint, what does it mean that the dependence between x and y is computable?
- For a multi-valued function, for the same input x, we may get several different values y.
- In this case, it is desirable to compute the *set* of all possible value y corresponding to a given x.
- For compact Y, the set of x-possible values of y is pre-compact.
- Thus, with any given accuracy, this set can be described by a finite list L of possible values:
  - if y is a possible value of $f(x)$, then y should be close to one of the values from L;
  - vice versa, each value from L should be close to some $f(x)$.

9. Additional Problem: Discontinuity

- Let us consider $f(x) = \text{sign}(x)$.
- At each stage of the computation, we only know an approximate value of x.
- So, even when actually $x = 0$, we cannot exclude that $x > 0$ or $x < 0$, so all 3 values $(0, \pm 1)$ are possible.
- In general, we need to take into account not only $f(x)$ but also $f(x')$ for close x'.
- In view of this, the above properties of the list L must be appropriately modified:
  - if y is a possible value of $f(x')$ for some $x' \approx x$, then y should be close to one of the values from L;
  - for every value from L, there must exist a close y which is a possible value of $f(x')$ for some $x' \approx x$.

10. Resulting Definition

- Let X and Y be computable compact sets with metrics d_X and d_Y.
- An everywhere defined relation $R \subseteq X \times Y$ is called *computable* if there exists an algorithm that:
  - given a computable element $x \in X$ and computable positive numbers $0 < \varepsilon < \varepsilon'$ and $0 < \delta$,
  - produces a finite list $\{y_1, \ldots, y_m\} \subseteq Y$

  such that:

  (1) if $(x', y) \in R$ for some x' for which $d_X(x', x) \leq \varepsilon$, then there exists an i for which $d_Y(y, y_i) \leq \delta$;

  (2) $\forall$ element y_i from this list, $\exists$ values x' and y for which $d_X(x, x') \leq \varepsilon'$, $d_Y(y_i, y) \leq \delta$, and $(x', y) \in R$.

11. Main Result

- Let X, Y be metric spaces with metrics d_X and d_Y.
- Their Cartesian product $X \times Y$ is the set of all pairs (x, y), $x \in X$ and $y \in Y$, with metric

$$d_{X\times Y}((x,y),(x',y')) \stackrel{\text{def}}{=} \max(d_X(x,x'), d_Y(y,y')).$$

- One can check that if X and Y are both compact sets, then the product $X \times Y$ is also a compact set.
- **Proposition.**
  - *Let X and Y be computable compact sets.*
  - *A relation $R \subseteq X \times Y$ is computable if and only if the set R is a computable compact set.*
- So, computability is equivalent to constructive compactness of the graph of R.

12. Inverse Relations: Corollary

- An inverse relation can be defined as

$$R^{-1} = \{(x, y) : (y, x) \in R\}.$$

- This is a natural generalization of the notion of an inverse function; for example:
  - $\pm\sqrt{x}$ is inverse to x^2:
  - $\ln(x)$ is inverse to $\exp(x)$;
  - $\arcsin(x)$ is inverse to $\sin(x)$.
- **Corollary.**
  - *If the range of R is the whole set Y,*
  - *then a multi-valued function (relation) R is computable if and only if its inverse is computable.*

13. Acknowledgments

This work was supported in part:

- by NSF grants HRD-1242122 and DUE-0926721,
- by NIH Grants 1 T36 GM078000-01 and 1R43TR000173-01, and
- by an ONR grant N62909-12-1-7039.

14. Idea of The Proof

$\Leftarrow$ Let R be a computable compact set, let x be a computable element of X, and let $\varepsilon < \varepsilon'$.

- Then, $\exists$ computable $\varepsilon_0 \in (\varepsilon, \varepsilon')$ s.t. $S \stackrel{\text{def}}{=} \{(x', y) \in R : d_X(x, x') \le \varepsilon_0\}$ is a computable compact.
- Thus, for a given computable $\delta > 0$, there exists a finite δ-net $L = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ for S.
- One can then prove that the list $\{y_1, \ldots, y_m\}$ satisfies the desired properties of a computable f-n for $f(x)$.

$\Rightarrow$ Vice versa, let R is a computable function.

- Since X is a compact, it has an α-net $\{x_1, \ldots, x_k\}$.
- For each i, we have a list $\{y_{i1}, \ldots, y_{im_i}\}$ which α-approximates $f(x_i)$.
- Then, the pairs (x_i, y_{ij}) form an α-net for the set R.