Algebraic Product is the Only t-Norm for Which Optimization Under Fuzzy Constraints is Scale-Invariant

Juan Carlos Figueroa Garcia[1], Martine Ceberio[2], and Vladik Kreinovich[2]

[1]Universidad Distrital, Departamento de Ingenieria Industrial
Bogota, Colombia, filthed@gmail.com
[2]University of Texas at El Paso, El Paso, Texas 79968, USA
mceberio@utep.edu, vladik@utep.edu

1. Need for Optimization under Fuzzy Constraints

- Example: we need to build a chemical plant for producing chemicals needed for space exploration.
- Among all designs x with small effect on environment, we need to select the most profitable one.
- Here, for each alternative x, we can compute the value $f(x)$ of the objective function.
- Constraints are formulated by using imprecise words from a natural language (like "small").
- In fuzzy logic, to each alternative x, we assign degree $\mu_c(x)$ to which x is, e.g., small.
- E.g., if a user marks smallness of x by 7 on a scale 0 to 10, we take $\mu_c(x) = 7/10$.
- Problem: find x such that $f(x) \to \max$ under constraint $\mu_c(x)$.

2. Bellman-Zadeh Approach to Optimization under Fuzzy Constraints

- First, we find the smallest value m of the objective function $f(x)$ among all possible solutions x.
- Then, we find the largest possible value M of the objective function over all possible constraints.
- We form the degree to which x is maximal:

$$\mu_m(x) \stackrel{\text{def}}{=} \frac{f(x) - m}{M - m}.$$

- We want to find an alternative which satisfies the constraints *and* maximizes the objective function.
- In fuzzy techniques, the degree of truth in "and"-statement is described by an appropriate *t-norm* $f(a, b)$.
- So, we select x for which the degree $\mu_s(x) = f_\&(\mu_c(x), \mu_m(x))$ is the largest.

3. Problem: the Value M is Not Well Defined

- Usually, we have experience with similar problems, so we know previously selected alternative(s) x.
- The value $f(x)$ for such "status quo" alternatives can be used as the desired minimum m.
- Finding M is much more complicated, we do not know which alternatives to include and which not to include.
- If we replace the original value M with a new value $M' > M$, then the maximizing degree changes:

$$\mu_m(x) = \frac{f(x) - m}{M - m} \to \mu'_m(x) = \frac{f(x) - m}{M' - m}.$$

- Here, $\mu'_m(x) = \lambda \cdot \mu_m(x)$ for $\lambda \stackrel{\text{def}}{=} \dfrac{M - m}{M' - m} < 1$.
- In general, diff. alternatives max $\mu_s(x) = f_\&(\mu_c(x), \mu_m(x))$ and $\mu'_s(x) = f_\&(\mu_c(x), \mu'_m(x)) = f_\&(\mu_c(x), \lambda \cdot \mu_m(x))$.

4. What We Show in This Talk

In this paper:

- we show that the dependence on M disappears if we use algebraic product t-norm $f_{\&}(a, b) = a \cdot b$.
- We also show that this is the only t-norm for which decisions do not depend on M.

5. Definitions

- *By a* t-norm, *we mean a f-n* $f_\& : [0,1] \times [0,1] \to [0,1]$ *s.t.* $f_\&(a,b) = f_\&(b,a)$ *and* $f_\&(1,a) = a$ *for all* a, b.
- *We say that optimization under fuzzy constraints is* scale-invariant *for the t-norm* $f_\&(a,b)$ *if*
  - *for every set* X, *for every two functions* $\mu_c : X \to [0,1]$ *and* $\mu_m : X \to [0,1]$, *and*
  - *for every real number* $\lambda \in (0,1)$,

  we have $S = S'$, *where:*

  - S *is the set of all* $x \in X$ *for which the function* $\mu_s(x) = f_\&(\mu_c(x), \mu_m(x))$ *attains its maximum;*
  - S' *is the set of all* $x \in X$ *for which the function* $\mu_s'(x) = f_\&(\mu_c(x), \lambda \cdot \mu_m(x))$ *attains its maximum.*

6. Main Results

- **Proposition 1.** *For the algebraic product t-norm $f_\&(a,b) = a\cdot b$, optimization under fuzzy constraints is scale-invariant.*
- **Proposition 2.** *$a \cdot b$ is the only t-norm for which optimization under fuzzy constraints is scale-invariant.*
- It is usually required that the t-norm is associative.
- Our result does not require associativity, so we can apply to non-associative and-operations.
- Such operations sometimes more adequately represent human reasoning (Zimmermann, Zysno).

7. Proof of Proposition 1

- For the algebraic product t-norm $f_\&(a,b) = a \cdot b$:
    - S is the set of all $x \in X$ for which the function $\mu_s(x) = \mu_c(x) \cdot \mu_m(x)$ attains its maximum, and
    - S' is the set of all $x \in X$ for which the function $\mu'_s(x) = \mu_c(x) \cdot \lambda \cdot \mu_m(x)$ attains its maximum.
- Here, $\mu'_s(x) = \lambda \cdot \mu_s(x)$ for a positive number λ.
- Clearly, $\mu_s(x) \ge \mu_s(y)$ if and only if $\lambda \cdot \mu_s(x) \ge \lambda \cdot \mu_s(y)$.
- So the optimizing sets S and S' for $\mu_s(x)$ and $\mu'_s(x) = \lambda \cdot \mu_s(x)$ indeed coincide.

8. Proof of Proposition 2

- Let $f_\&(a,b)$ be a t-norm for which optimization under fuzzy constraints is scale-invariant.
- Let $a, b \in [0,1]$; let us prove that $f_\&(a,b) = a \cdot b$.
- Let us consider $X = \{x_1, x_2\}$ with $\mu_c(x_1) = \mu_m(x_2) = a$ and $\mu_c(x_2) = \mu_m(x_1) = 1$.
- Here, $\mu_s(x_1) = f_\&(\mu_c(x_1), \mu_m(x_1)) = f_\&(a,1) = a$.
- Similarly, $\mu_s(x_2) = f_\&(\mu_c(x_2), \mu_m(x_2)) = f_\&(1,a) = a$.
- Since $\mu_s(x_1) = \mu_s(x_2)$, the optimizing set S consists of both elements x_1 and x_2.
- Due to scale-invariance, for $\lambda = b$, $S' = S = \{x_1, x_2\}$ is optimizing for $\mu'_s(x) = f_\&(\mu_c(x), \lambda \cdot \mu_m(x))$.
- Thus, $\mu'_s(x_1) = \mu'_s(x_2)$, i.e., $f_\&(a, b \cdot 1) = f_\&(1, b \cdot a)$.
- So, $f_\&(a,b) = f_\&(1, b \cdot a) = a \cdot b$. Q.E.D.

9. Acknowledgments

- This work was supported in part:
  - by the National Science Foundation grants 0953339, HRD-0734825, HRD-1242122, DUE-0926721,
  - by Grants 1 T36 GM078000-01 and 1R43TR000173-01 from the National Institutes of Health, and
  - by grant N62909-12-1-7039 from the Office of Naval Research.
- This work was performed when J. C. Figueroa Garcia was a visiting researcher at Univ. of Texas at El Paso.

10. Bibliography

- R. E. Bellman and L. A. Zadeh, "Decision making in a fuzzy environment", *Management Science*, 1970, Vol. 47, No. 4, pp. B141–B145.
- L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.
- H. H. Zimmerman and P. Zysno, "Latent connectives in human decision making", *Fuzzy Sets and Systems*, 1980, Vol. 4, pp. 37–51.