How to Faster Test a Device for Different Combinations of Parameters

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1. Formulation of the Problem

- Many devices have to function correctly under many different values of the corresponding parameters: e.g.,
  - for temperatures within the given range,
  - for pressure within the given range,
  - for humidity within the given range, etc.
- Ideally, we should test the device for all possible combinations of the corresponding parameters.
- However, often, such a testing is not realistic. For example:
  - if we have 20 possible parameters, and
  - we consider 10 possible values of each of these parameters,
  - then 10^{20} tests require $3 \cdot 10^{12}$ years – longer than the lifetime of the Universe.

2. Solution: Test for All Pairs, or All Triples, etc.

- We cannot test for all possible combinations of all the parameters.
- So, we need to test at least for all possible values of each parameter separately:
  - for all possible values of outside temperature,
  - for all possible values of humidity, etc.
- In this testing, we may overlook possible joint effect of two or more different parameters.
- To take such an effect into account, it makes sense to test all *pairs* of values.
- Similarly, we may want to test all possible *triples* of values, etc.

3. How to Arrange Such a Test: First Simple Idea

- Let us assume that for each of n parameters, we test for N different values.
- In this case, we need $n \cdot N$ experiments to test the device's behavior for all N values of each of n parameters.
- For each of $\binom{n}{2}$ *pairs* of parameters, we test all possible N^2 pairs of values.
- Thus, we need $\binom{n}{2} \cdot N^2$ experiments.
- For each of $\binom{n}{k}$ *k-tuples* of parameters, we test all possible N^k tuples of values.
- Thus, we need $\binom{n}{k} \cdot N^k$ experiments.

4. We Can Test Faster than That

- To test all possible values of each parameter, the above approach requires $n \cdot N$ experiments.
- In reality, it is sufficient to perform only N experiments:
  - in the first experiment, we select the first value of each of n parameters;
  - in the second experiment, we select the second value of each of n parameters; etc.
- When we have many parameters $n \gg 1$, we then have $n \cdot N \gg N$.
- So, this idea drastically decreases the number of necessary experiments – and thus, the testing time.
- We show that a similar speed-up is possible when we test all possible pairs (triples, etc.) of parameters.

5. Formulating the Problem in Precise Terms

- *Let n, N, and k be positive natural numbers.*
- *The number n will be called the* number of parameters, *and the number N will be called the* number of values.
- *By an* experiment, *we mean a tuple of n integers $j_1, \ldots, j_n$, where $1 \leq j_i \leq N$ for all i.*
- *By a* testing design, *we mean a finite set of experiments.*
- *We say that a testing design T* tests each combination of k parameters *if:*
  - *for every two k-tuples $1 \leq i_1 < \ldots < i_k \leq N$ and $v_1, \ldots, v_k$ $(1 \leq v_\ell \leq N)$,*
  - *T contains an experiment in which we use the v_ℓ-th value of each i_ℓ-th parameter.*

6. Main Result

- *Objective:* minimize the number of experiments.
- *Naive idea:* tests each of $\binom{n}{k} \cdot N^k = O(n^k) \cdot N^k$ combinations of k parameters.
- For $n = k$, we need to test all N^k possible combinations of parameters, so we cannot have fewer than N^k tests.
- However, as the above case of $k = 1$ shows, we can try to minimize the factor depending on n.
- *For each k, there is a design that tests each combination of k parameters in $O(\log^{k-1}(n)) \cdot N^k$ experiments.*
- For $k = 1$, we get the known fact that we need $O(N)$ experiments.
- For testing all pairs ($k = 2$), we need $O(\log(n)) \cdot N^2$ experiments ($\ll O(n^2) \cdot N^2$ experiments in naive case).

7. New Testing Design: Case $k = 2$

- Our design consists of $B = \lceil \log_2(n) \rceil \sim \log(n)$ groups, each having N^2 experiments.
- Each number $x \le n - 1$ can be represented by B bits $\text{bit}_j(x)$, $j = 1, \ldots, B$.
- In the b-th group, for each pair of integers (f, s) such that $1 \le f, s \le N$, we take:
  - $j_i = f$ if $\text{bit}_b(i-1) = 0$, and
  - $j_i = s$ if $\text{bit}_b(i-1) = 1$.
- For each pair $i_1 < i_2$, at least one bit b in the binary expansions of $i_1 - 1$ and $i_2 - 1$ is different.
- For this bit b, the corresponding group of experiments tests all possible pairs (f, s).

8. First Example: $n = 2$

- For $n = 2$, we need $B = 1$ bit to represent integers 0 and 1.
- Thus, in this case, we have a single group of experiments: for each pair (s, f), we set $x_1 = f$ and $x_2 = s$.
- In other words, each experiment has the form (s, f).

9. Second Example: $n = 4$

- For $n = 4$, we need $B = 2$ bits to represent 0, 1, 2, and 3: $0_{10} = 00_2$, $1_{10} = 01_2$, $2_{10} = 10_2$, $3_{10} = 11_2$.
- In this case, we have two groups of N^2 experiments.
- In the 1st group, we assign s to all i s.t. $\text{bit}_1(i-1) = 0$, and f to all i s.t. $\text{bit}_1(i-1) = 1$.
- Thus, each experiment has the form (f, s, f, s).
- In the 2nd group, we assign s to all i s.t. $\text{bit}_2(i-1) = 0$, and f to all i s.t. $\text{bit}_2(i-1) = 1$.
- Thus, each experiment has the form (f, f, s, s).
- If $i_1 < i_2$ are both odd or even, then the 2nd group of experiments tests all possible combinations of values.
- If one of i_1 and i_2 is odd and another even, then the 1st group tests all possible combinations of values.

10. Third Example: $n = 8$

- For $n = 8$, we need $B = 3$ bits to represent integers from 0 to 7.
- Thus, in this case, we have three groups of N^2 experiments each.
- In the first group of experiments, each experiment has the form
$$(f, s, f, s, f, s, f, s).$$
- In the second group of experiments, each experiment has the form
$$(f, f, s, s, f, f, s, s).$$
- In the third group of experiments, each experiment has the form
$$(f, f, f, f, s, s, s, s).$$

11. Testing Design: Case of $k > 2$

- For $n = k$, we just have to test all N^k possible combinations of values of all k parameters.
- For $n > k$, we divide the set of n parameters into two halves of size $n/2$.
- To cover situations when all k parameters are in the 1st half, we use the testing design for $n/2$ and k.
- Each experiment in this design is copied for the second half, so, e.g., a design fs becomes $fsfs$.
- To cover situations in which i parameters are in the 1st half, we combine:
  - each experiment from design for $n/2$ and i with
  - each experiment from design for $n/2$ and $k - i$.

12. Case of $k > 2$: Number of Experiments

- Let $E_k(n)$ be the number of experiments that our algorithm requires for n and k; then:

$$E_k(n) = E_k(n/2) + \sum_{i=1}^{k-1} E_{k-i}(n/2) \cdot E_i(n/2).$$

- One can prove, by induction, that this implies that

$$E_k(n) = O(\log_2^k(n)) \cdot N^k.$$

13. Example: $n = 4$ and $k = 3$

- It is not possible to have $n/2 = 2$ parameters and test all possible values of $k = 3$ of them.
- We combine each experiment with $n = 2$ and $k = 2$ with each experiment with $n = 2$ and $k = 1$.
- There is one group of experiments with $n = 2$ and $k = 2$: sf, with s and f going from 1 to N.
- There is one group corresponding to $n = 2$ and $k = 1$: tt, with t from 1 to N.
- Thus, by combining them, we get experiments of the type $sftt$.
- Finally, we combine each experiment with $n = 2$ and $k = 1$ with each experiment with $n = 2$ and $k = 2$.
- Similarly, we get $ttsf$.
- So, we get two groups of experiments: $sftt$ and $ttsf$.

14. Example: $n = 8$ and $k = 3$

- First, we list all the experiments corresponding to $n/2 = 4$ and $k = 3$, and repeat each for the second half as well.
- Thus, from $sftt$ and $ttsf$, we get $sfttsftt$ and $ttsfttsf$.
- Then, we combine each experiment with $n = 4$ and $k = 2$ with each experiment with $n = 4$ and $k = 1$.
- There are 2 groups of experiments with $n = 4$ and $k = 2$: $sfsf$ and $ssff$.
- There is one group for $n = 4$ and $k = 1$: $tttt$.
- Combining, we get $sfsftttt$ and $ssfftttt$.
- Finally, we combine each experiment with $n = 4$ and $k = 1$ with each experiment with $n = 4$ and $k = 2$.
- Combining, we get $ttttsfsf$ and $ttttssff$.
- Totally, we have 6 groups of N^3 experiments.