From Global to Local Constraints: A Constructive Version of Bloch's Principle

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1. Outline

- Generalizing several results from complex analysis, A. Bloch formulated an informal principle:
  - for every global implication
  - there is a stronger local implication.
- This principle has been formalized for complex analysis.
- However, is has been successfully used in other areas as well.
- In this talk, we propose a new formalization of Bloch's Principle.
- We also show that in general, the corresponding localized version can be obtained algorithmically.

2. Bloch's Principle: A Brief History

- *Liouville's Theorem:* if $f(z)$ is analytical, bounded on a complex plane, and $f(0) = 0$, then $f(z) \equiv 0$.
- This theorem requires that the constraint $|f(z)| \le C$ be satisfied *globally*, i.e., for all z.
- What if this constraint is only satisfied *locally*, i.e., for all z from a bounded set? *H. A. Schwarz's theorem*:
  - if $f(z)$ is analytical, $f(0) = 0$, and $|f(z)| \le C$ for all $|z| \le R$,
  - then for all $z \in B_R(0) \stackrel{\text{def}}{=} \{z : |z| \le R\}$, we have $|f(z)| \le \dfrac{C}{R} \cdot |z|$.
- When the size R increases, the bound tends to 0; so for $R \to \infty$, we get Liouville's Theorem.

3. Bloch's Principle (cont-d)

- Several similar localizations of global results are known.
- In 1926, A. Bloch, formulated a general (informal) *Bloch's Principle*:
  - for every global result,
  - there is a local version from which this global result follows.
- In complex analysis, this principle was formalized.
- However, there are many interesting results of the use of Bloch's Principle in other areas of mathematics.
- Can we formalize Bloch's Principle in a context which is more general than complex analysis?
- If yes, and if the appropriate the localization always exists, can we find it algorithmically?

4. Analysis of the Problem

- The Liouville's Theorem (LT) has the form

  $$\forall f \in \mathcal{F}\,(\forall x\,(f(x) \in A(x)) \Rightarrow \forall x\,(f(x) \in B(x))),\ \text{where:}$$

  - $\mathcal{F} = \{f : f \text{ is analytical} \,\&\, f(0) = 0\}$,
  - $A(x) = \{x : |x| \le C\}$ and $B(x) = \{0\}$.
- *LT:* if the constraint $f(x) \in A(x)$ is *exactly* satisfied for *all* values x, then the conclusion holds.
- *Localized version:*
  - when the constraint is "approximately" satisfied – with some accuracy $\delta > 0$ for all $x \in B_r(0)$,
  - then the conclusion is also approximately satisfied, with some accuracy $\varepsilon > 0$ and for $x \in B_R(0)$.
- When $\delta \to 0$ and $r \to \infty$, then $\varepsilon \to 0$ and $R \to \infty$.

5. Analysis of the Problem (cont-d)

- $z \in S \Leftrightarrow d(z, S) \stackrel{\text{def}}{=} \inf\{d(z, s) : s \in S\} = 0$ for compact S, so LT is equivalent to

$$\forall f\, (\forall x\, d(f(x), A(x)) = 0 \Rightarrow \forall x\, d(f(x), B(x)) = 0)).$$

- *Localized version:* $\forall \varepsilon > 0\, \forall R > 0\, \exists \delta > 0\, \exists r > 0$

$$\forall f\, ((\forall x\, (d(x, x_0) \le r \Rightarrow d(f(x), A(x)) \le \delta)) \Rightarrow$$

$$(\forall x\, (d(x, x_0) \le R \Rightarrow d(f(x), B(x)) \le \varepsilon))).$$

- $A(x)$ and $B(x)$ are continuous in x – in Hausdorff metric $d_H(A, B) \stackrel{\text{def}}{=} \max\left(\max\limits_{a \in A} d(a, B), \max\limits_{b \in B} d(b, A)\right)$.

- For every bounded set D, limitations of $\mathcal{F}$ to D form a compact set.

- $\mathcal{F}$ is *locally defined*: if $f(x)$ coincides with some analytical f-n in every neighborhood, then $f(x)$ is analytical.

6. Definitions

- A metric space X is called *bounded-compact* (b.c.) if every closed bounded set in X is compact.
- A class of f-ns $\mathcal{F}$ from b.c. X to b.c. Y is *bounded-compact* (b.c.) if for every compact $K \subset X$,

$$\mathcal{F} \text{ is compact in } d_K(f, g) \stackrel{\text{def}}{=} \sup_{x \in K} d(f(x), g(x)).$$

- A f-n $f : X \to Y$ locally belongs to $\mathcal{F}$ if for every n, there exists a f-n $f_n \in \mathcal{F}$ coinciding with f on

$$B_n(x_0) = \{x : d(x, x_0) \leq n\}.$$

- *Comment:* this notion does not depend on x_0.
- A b.c. class $\mathcal{F}$ is *locally defined* (l.d.) if it contains all the functions that locally belong to $\mathcal{F}$.
- Let $\mathcal{F}$ be b.c. and l.d. An $\mathcal{F}$*-constraint* A is a continuous f-n mapping $x \in X$ into a compact $A(x) \subseteq Y$.

7. Main Result

- Let $\mathcal{F}$ be a bounded-compact locally defined class of functions, and let A and B be $\mathcal{F}$-constraints.
  - We say that the A *globally implies* B if for every $f \in \mathcal{F}$, $\forall x\,(f(x) \in A(x))$ implies $\forall x\,(f(x) \in B(x))$.
  - A *locally implies* B if $\forall \varepsilon > 0\, \forall R > 0\, \exists \delta > 0\, \exists r > 0$:

$$\forall f\,((\forall x\,(d(x, x_0) \le r \Rightarrow d(f(x), A(x)) \le \delta)) \Rightarrow$$

$$(\forall x\,(d(x, x_0) \le R \Rightarrow d(f(x), B(x)) \le \varepsilon))).$$

- *Proposition:*
  – Let $\mathcal{F}$ be a bounded-compact locally defined class of functions, and let A and B are $\mathcal{F}$-constraints.
  – If A globally implies B, then A locally implies B.

8. Proof by Contradiction: Main Ideas

- If A does not locally imply B, then there exist $\varepsilon > 0$ and $R > 0$ s.t. $\forall n\, \exists f_n \in \mathcal{F}\, \exists x_n \in B_R(x_0)$ s.t.
$$\max_{x \in B_n(x_0)} d(f_n(x), A(x)) \le \frac{1}{n} \,\&\, d(f_n(x_n), B(x_n)) > \varepsilon.$$
- Since $x_n \in B_R(x_0)$, it has a convergent subsequence.
- W.l.o.g., we can assume that $x_n \to \ell$.
- Since $\mathcal{F}$ is compact relative to $d_{B_k(x_0)}$, from f_n, we can extract a subsequence $n(1,i)$ convergent for $k = 1$.
- From this subsequence, we can similarly extract a subsequence $n(2,i)$ which is convergent for $k = 2$, etc.
- Diagonal subsequence $f_{n(i,i)}$ then converges for all k.
- This convergence is for all x, so we can defining a pointwise limit function $f(x)$.

9. Proof (cont-d)

- For each x, we can define a point-wise limit f-n $f(x)$.
- On each ball $B_k(x_0)$, this limit coincides with the corresponding limit from $\mathcal{F}$ limited to this ball.
- Thus, $f(x)$ locally belongs to $\mathcal{F}$.
- Since $\mathcal{F}$ is locally defined, $f \in \mathcal{F}$.
- For f, for every x, $d(f_n(x), A(x)) \le \dfrac{1}{n}$ tends to
$$d(f(x), A(x)) = 0.$$
- Since A globally implies B, we conclude that we have $d(f(x), B(x)) = 0$ for all x.
- In particular, we have $d(f(\ell), B(\ell)) = 0$.
- However, from $d(f_n(x_n), B(x_n)) > \varepsilon$, in the limit $x_n \to \ell$, we get $d(f(\ell), B(\ell)) \ge \varepsilon > 0$; a contradiction.

10. Bloch's Principle: A Constructive Version

- Let spaces X and Y be computable and computably bounded-compact.
- Let A and B be computable functions for which A globally implies B.
- Then there exists an algorithm that:
  - *given* rational numbers $\varepsilon > 0$ and $R > 0$,
  - *produces* computable numbers $\delta > 0$ and $r > 0$ for which A locally implies B for ε, R, δ, and r, i.e.,

$$\forall f\,((\forall x\,(d(x,x_0) \le r \Rightarrow d(f(x),A(x)) \le \delta)) \Rightarrow$$

$$(\forall x\,(d(x,x_0) \le R \Rightarrow d(f(x),B(x)) \le \varepsilon))).$$

11. Proof of Constructive Bloch's Principle

- We know that for $\varepsilon_0 = \varepsilon/3$ and for $R_0 = R+1$, $\exists n = n_0$ s.t. $r = n$ and $\delta = 1/n$ satisfy the desired property.
- Let us show that we can find this n by checking whether $n = 1, 2, \ldots$ satisfy the desired condition.
- There are known algorithms for computing max and min of a computable f-n over a computable compact.
- *Known:* for a computable function $F(x)$ on a computable compact K:
  - for every two computable numbers $z^- < z^+$ within the range of $F(x)$ on K,
  - we can compute a $z \in (z^-, z^+)$ s.t. $\{x : F(x) \le z\}$ is a computable compact.
- We thus compute $R' \in (R, R+1)$ s.t. $B_{R'}(x_0)$ is computably compact.

12. Algorithm (cont-d)

- For each n, we similarly compute $r_n \in (n-1, n)$ s.t. $B_{r_n}(x_0)$ is a computable compact.
- Since the ball $B_{r_n}(x_0)$ is a computable compact, the value $v(f) \stackrel{\text{def}}{=} \max\limits_{x \in B_{r_n}(x_0)} d(f(x), A(x))$ is also computable.
- So, $v(f)$ a computable function of $f \in \mathcal{F}' \stackrel{\text{def}}{=} \mathcal{F}_{|B_{x_0}(R)}$.
- The restriction $\mathcal{F}'$ is a computable compact.
- Thus, we can compute $\delta_n \in (1/n, 1/(n-1))$ s.t. $S \stackrel{\text{def}}{=} \{f : v(f) \le \delta_n\}$ is a computable compact.
- We can therefore compute $M \stackrel{\text{def}}{=} \max\limits_{x \in B_{R'}(x_0),\ f \in S} d(f(x), B(x))$ with accuracy $\dfrac{\varepsilon}{3}$.
- If the resulting estimate $\widetilde{M}$ is $\le \dfrac{2}{3} \cdot \varepsilon$, we stop.

13. When We Stop, We Get the Desired Values: Proof

- Let us assume that for some f, for all $x \in B_n(x_0)$, we have $d(f(x), A(x)) \le 1/n < \delta_n$.
- Since $r_n < n$, this inequality is also true for all
$$x \in B_{r_n}(x_0).$$
- Hence, for $v(f) = \max\limits_{x \in B_{r_n}(x_0)} d(f(x), A(x))$, we get
$$v(f) < \delta_n.$$
- Every $x \in B_R(x_0)$ belongs to $B_{R'}(x_0)$ and thus, for this x, we have $d(f(x), B(x)) \le M$.
- Since $M = \max\limits_{x \in B_{R'}(x_0),\ f \in S} d(f(x), B(x)) \le \widetilde{M} + \dfrac{\varepsilon}{3}$ and $\widetilde{M} \le \dfrac{2}{3} \cdot \varepsilon$, we get $M \le \varepsilon$, hence the desired inequality
$$d(f(x), B(x)) \le \varepsilon.$$

14. Algorithm Stops for $n = n_0 + 1$

- By def-n of n_0, if $x \in B_{n_0}(x_0)$ and $d(f(x), A(x)) \le \dfrac{1}{n_0}$, then $d(f(x), B(x)) \le \varepsilon/3$ for all $x \in B_{R_0}(x_0)$.
- $R' < R + 1 = R_0$, so $x \in B_{R'}(x_0)$ implies $x \in B_{R_0}(x_0)$.
- Similarly, since $r_n > n - 1 = n_0$, we conclude that

$$\max_{x \in B_{n_0}(x_0)} d(f(x), A(x)) \le v(f) = \max_{x \in B_{r_n}(x_0)} d(f(x), A(x)).$$

- So, $v(f) \le \delta_n < \dfrac{1}{n_0} \Rightarrow \max\limits_{x \in B_{n_0}(x_0)} d(f(x), A(x)) < \dfrac{1}{n_0}$.
- Thus, for all such x and f, we have $d(f(x), B(x)) \le \dfrac{\varepsilon}{3}$.
- Hence, $M = \max\limits_{x \in B_{R'}(x_0),\ f \in S} d(f(x), B(x)) \le \dfrac{\varepsilon}{3}$.
- So $\widetilde{M} \le M + \dfrac{\varepsilon}{3} \le \dfrac{2}{3} \cdot \varepsilon$, and the algorithm will stop.

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16. Bibliography

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