Range Estimation under Constraints is Computable Unless There Is a Discontinuity

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1. Outline

- One of the main problems of interval computations is computing the range of a given f-n over given intervals.
- There is a general algorithm for computing the range of computable functions over computable intervals.
- However, in practice, not all possible combinations of the inputs are possible, i.e., there are constraints.
- Under constraints, it becomes impossible to have an algorithm which would always compute this range.
- In this talk, we explain that the main reason why range estimation under constraints is not always computable:
  - constraints may introduce discontinuity, while
  - all computable functions are continuous.
- We show that under computably continuous constraints, the problem of range estimation remains computable.

2. Need for Data Processing

- We often need to make a decision, e.g., to select an engineering design and/or control strategy.
- For this, we need to know the effects of selecting different alternatives.
- In most engineering problems, we know:
  - how different quantities depend on each other and
  - how they change with time.
- In particular, we usually know the dependence
  - of the quantity y describing the effect
  - on the values of the quantities $x_1, \ldots, x_n$ describing the decision and the surrounding environment:

$$y = f(x_1, \ldots, x_n).$$

- The resulting computations are known as *data processing*.

3. Need to Take Uncertainty into Account

- In the ideal situation, we know the exact values $x_1, \ldots, x_n$ of the corresponding parameters.
- Then, we can simply substitute these values into a known function f, and get the desired value y.
- In practice, the values $x_1, \ldots, x_n$ come from measurements, which are never absolutely accurate.
- The measurement results $\widetilde{x}_1, \ldots, \widetilde{x}_n$ are, in general, somewhat different from the actual (unknown) values x_i.
- Thus, the estimate $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ is, in general, different from the desired value $y = f(x_1, \ldots, x_n)$.
- To make an appropriate decision, it is important to know how big can be the difference $\widetilde{y} - y$.

4. Need for Range Estimation

- Often, our only information about the measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ is the upper *bound* Δ_i: $|\Delta x_i| \le \Delta$.
- In this case, based on the measurement result $\widetilde{x}_i$, we only know that $x_i \in [\underline{x}_i, \overline{x}_i] \stackrel{\text{def}}{=} [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.
- Another case of such an interval uncertainty is when the parameter x_i characterizes a manufactured part.
- In this case, we know that the corresponding value must lie within the *tolerance interval* $[\underline{x}_i, \overline{x}_i]$.
- Different values $x_i \in [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$ lead, in general, to different values of $y = f(x_1, \ldots, x_n)$.
- It is therefore important to estimate the *range* of all such values $\{f(x_1, \ldots, x_n) : x_i \in [\underline{x}_i, \overline{x}_i] \text{ for all } i\}$.

5. Interval Computations

- In the usual case of continuous f-ns f, this range is an interval $[\underline{y}, \overline{y}] \stackrel{\text{def}}{=} \{f(x_1, \ldots, x_n) : x_i \in [\underline{x}_i, \overline{x}_i] \text{ for all } i\}$.
- Estimation of this range interval is known as *interval computations*.
- For computable functions f on computable intervals $[\underline{x}_i, \overline{x}_i]$, there is an algorithm which computes this range.
- In general, the corresponding computational problem is NP-hard (i.e., it may take a very long time).
- However, there are many situations where feasible algorithms are possible for exact computations.
- There are also many feasible algorithms for providing *enclosures* for the desired ranges.

6. Need to Take Constraints into Account

- The above formulation of range estimation problem assumes that the quantities x_1, …, x_n are independent:
  - the set of possible values of, e.g., x_1,
  - does not depend on the actual values of all other quantities.
- In practice, we often have additional *constraints* which limit possible combinations of values $(x_1, \ldots, x_n)$.
- For example, if x_1 and x_2 represent the control values at two consequent moments of time, then

$$|x_1 - x_2| < \delta \text{ for some small value } \delta > 0.$$

- We are interested in the range of the values $f(x_1, \ldots, x_n)$ corr. to $(x_1, \ldots, x_n)$ satisfying all the constraints.
- Adding constraints makes the general problem not computable.

7. What Is Computable: Reminder

- A real number x is *computable* if there exists an algorithm that, given $k \in N$, returns a rational r_k s.t.

$$|r_k - x| \le 2^{-k}.$$

- An interval $[\underline{x}, \overline{x}]$ is called *computable* if both its endpoints are computable.

- A function $f(x_1, \ldots, x_n)$ from real numbers to real numbers is called *computable* if there exist two algorithms:

  – an algorithm that, given rational numbers $r_1, \ldots, r_n$, and an integer k, returns a rational number r s.t.

$$|r - f(r_1, \ldots, r_n)| \le 2^{-k};$$

  – an algorithm that, given a rational number $\varepsilon > 0$, returns a rational number $\delta > 0$ such that

$$\text{if } |x_i - x'_i| \le \delta \text{ for all } i, \text{ then } |f(x_1, \ldots, x_n) - f(x'_1, \ldots, x'_n)| \le \varepsilon.$$

8. Known Positive Result

- The following algorithm,
  - given a computable function $f(x_1, \ldots, x_n)$ and computable intervals $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$ $(1 \le i \le n)$,
  - returns the range $[\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_i \in \mathbf{x}_i\}$.
- We want to compute $\overline{y}$ with accuracy $\varepsilon > 0$.
- We find $\delta > 0$ s.t. $|x_i - x_i'| \le \delta$ implies that the values of f are $(\varepsilon/2)$-close to each other.
- On each interval $[\underline{x}_i, \overline{x}_i]$, we then select finitely many points $\underline{x}_i$, $\underline{x}_i + \delta$, $\underline{x}_i + 2\delta$, ...
- For each combination of selected points, we produce a rational r which is $(\varepsilon/2)$-close to $f(s_1, \ldots, s_n)$.
- The largest $\overline{r}$ of these rational numbers is the desired ε-approximation to $\overline{y}$.
- The smallest $\underline{r}$ of these r's is ε-close to $\underline{y}$.

9. Proof that the Above Algorithm Is Correct

- Each rational r is bounded by $f(s_1, \ldots, s_n) + \dfrac{\varepsilon}{2}$.
- Thus, $f(s_1, \ldots, s_n) \le \overline{y}$ implies $r \le \overline{y} + \dfrac{\varepsilon}{2}$.
- In particular, $\overline{r} \le \overline{y} + \dfrac{\varepsilon}{2} \le \overline{y} + \varepsilon$.
- Let us consider the values x_i s.t. $f(x_1, \ldots, x_n) = \overline{y}$.
- Each x_i is δ-close to some s_i, so:

$$|f(s_1, \ldots, s_n) - f(x_1, \ldots, x_n)| \le \frac{\varepsilon}{2} \text{ and } f(s_1, \ldots, s_n) \ge \overline{y} - \frac{\varepsilon}{2}.$$

- For the corresponding r, we have $r \ge f(s_1, \ldots, s_n) - \dfrac{\varepsilon}{2}$ and hence, $r \ge \overline{y} - \varepsilon$.
- Thus, $\overline{r} \ge r \ge \overline{y} - \varepsilon$.
- We have proved that $\overline{r} \le \overline{y} + \varepsilon$, so $\overline{r}$ is ε-close to $\overline{y}$.
- Similarly, $\underline{r}$ is ε-close to $\underline{y}$.

10. Range Estimation Under Constraints: A Problem

- Let $g_j(x_1, \ldots, x_n)$ be a computable function and c_j, $\underline{c}_j$, and $\overline{c}_j$ be computable numbers.
- By a *computable constraint*, we mean a constraint of one of the following types:
  - $g_j(x_1, \ldots, x_n) = c_j$,
  - $g_j(x_1, \ldots, x_n) \le c_j$,
  - $c_j \le g_j(x_1, \ldots, x_n)$, or
  - $\underline{c}_j \le g_j(x_1, \ldots, x_n) \le \overline{c}_j$.
- *Given:* a computable f-n $f(x_1, \ldots, x_n)$, computable intervals $[\underline{x}_i, \overline{x}_i]$, and a list C of computable constraints.
- *Compute:* $\overline{y} = \max\{f(x_1, \ldots, x_n) : x_i \text{ satisfy } C\}$ and $\underline{y} = \min\{f(x_1, \ldots, x_n) : x_i \text{ satisfy } C\}$.

11. Known Negative Result

- *Result:* no algorithm is possible which solves all the problems of range estimation under constraints.
- Indeed, take $n = 1$, $f(x_1) = x_1$, and a constraint $g(x_1) = c_1$, where $g(x_1) \stackrel{\text{def}}{=} \min(x_1, \max(0, x_1 - 1))$.
- For $x_1 \le 0$, we get $g(x_1) = x_1$; for $0 \le x_1 \le 1$, we get $g(x_1) = 0$, and for $x_1 \ge 1$, we get $g(x_1) = x_1 - 1$.
- For $c_1 < 0$, the constraint is only satisfied for the value $x_1 = c_1$, so we get $\overline{y} = c_1$.
- For $c_1 = 0$, the constraint $g(x_1) = c_1 = 0$ is satisfied for all $x_1 \in [0, 1]$, so we get $\overline{y} = 1$.
- So, the dependence of $\overline{y}$ on c_1 is discontinuous at $c_1 = 0$.
- However, all computable functions are continuous; Q.E.D.
- *We will prove* that discontinuity is the only obstacles to computing $\overline{y}$ and $\underline{y}$.

12. Computably Continuous Constraints: Definitions

- *Given:* computable f-s $g_j(x_1, \ldots, x_n)$, computable intervals $\mathbf{x}_i$, and constraint types $(=, \le, \ge, \le \cdot \le)$.
- For each tuple c of values c_j, $\underline{c}_j$, $\overline{c}_j$, we denote

$$S(c) \stackrel{\text{def}}{=} \{(x_1, \ldots, x_n) : x_i \in \mathbf{x}_i \text{ and } x_i \text{ satisfy the constraints}\}.$$

- The set of constraints is *computably continuous* if there is an algorithm that:
  - given a rational $\varepsilon > 0$,
  - returns a rational $\delta > 0$ s.t. when c and c' are δ-close, then $d_H(S(c), S(c')) \le \varepsilon$, where

$$d_H(A, B) \stackrel{\text{def}}{=} \max\left(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right), \text{ and}$$

$$d(a, B) \stackrel{\text{def}}{=} \inf_{b \in B} d(a, b).$$

13. Main Result

- The following algorithm solves the range estimation problem for all computably continuous constraints.
- We want to estimate $\underline{y}$ and $\overline{y}$ with accuracy ε.
- First, we find $\delta > 0$ for which $|x_i - x'_i| \le \delta$ implies that the f-values are ε-close.
- One can then show that if $d_H(S, S') \le \delta$, then $\max\limits_{x \in S} f(x)$ and $\max\limits_{x \in S'} f(x)$ are ε-close.
- For this $\delta > 0$, we can find $\beta > 0$ for which if c and c' are β-close, then $d_H(S(c), S(c')) \le \delta$.
- We can now replace each equality $g_j = c_j$ with inequalities $\underline{c}_j \le g_j \le \overline{c}_j$.
- As long as $|\underline{c}_j - c_j| \le \beta$ and $|\overline{c}_j - c_j| \le \beta$, we still have a δ-close set $S(c)$.

14. Main Result (cont-d)

- The box $[\underline{x}_1, \overline{x}_1] \times \ldots$ is a computable compact set.
- Due to the known property of such sets, there are β-close values c' for which $S(c')$ is a computable compact.
- Thus, the maximum $\overline{y}'$ and the minimum $\underline{y}'$ of the computable function $f(x)$ over $S(c')$ are computable.
- By choice of β, the fact that c' and c are β-close implies that $S(c')$ is δ-close to $S(c)$.
- Hence, for max and min over these sets, we have

$$|\overline{y}' - \overline{y}| \le \varepsilon \text{ and } |\underline{y}' - \underline{y}| \le \varepsilon.$$

- Thus, the computed values $\underline{y}'$ and $\overline{y}'$ are indeed ε-approximations to $\underline{y}$ and $\overline{y}$.

15. Auxiliary Result

- Let us consider the case when there are no equality constraints, and for two-sided inequalities, $\underline{c}_i < \overline{c}_i$.
- In this case, we can solve all problems of range estimation for which the dependence $S(c)$ is continuous.
- *Note:* $S(c)$ is not necessarily computably continuous.
- For $\beta = 2^{-k}$, $k = 0, 1, \ldots$, estimate the ranges of f:
  - $[\underline{y}', \overline{y}']$ over an inner β-approximation $S(c')$ and
  - $[\underline{y}'', \overline{y}'']$ over the outer β-approximations $S(c'')$.
- Then $\underline{y}'' \le \underline{y} \le \underline{y}'$ and $\overline{y}' \le \overline{y} \le \overline{y}''$.
- Due to continuity, $S(c')$ and $S(c'')$ will eventually become δ-close; then, $\underline{y}'$ and $\underline{y}''$ will be ε-close.
- When this happens, we return, e.g., $\underline{y}'$ and $\overline{y}'$ as the desired ε-approximations to $\underline{y}$ and $\overline{y}$.

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