Why Decimal System and Binary System Are the Most Widely Used: A Possible Explanation

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1. Outline

- What is so special about numbers 10 and 2 that decimal and binary systems are the most widely used?
- One interesting fact about 10 is that:
  - when we start with a unit interval and we want half-width, then this width is exactly $5/10$;
  - when we want to find a square of half area, its sides are almost exactly $7/10$, and
  - when we want to construct a cube of half volume its sides are almost exactly $8/10$.
- We show that $b = 2$, 4, and 10 are the only numbers with this property – at least when $b \le 10^9$.
- This may be a possible explanation of why decimal and binary systems are the most widely used.

2. Formulation of the Problem

- What is so special about numbers 10 and 2 that decimal and binary systems are the most widely used?
- This questions was raised, e.g., by Donald Knuth in his famous book *Art of Computer Programming*.
- One interesting fact about 10 is the following; when:
  - we start with a unit interval and
  - we want to constrict an interval of half width,
  - then this width is exactly $1/2 = 5/10$.
- When:
  - we start with a unit square and
  - we want to find a square of area $1/2$,
  - its sides are $\sqrt{1/2}$, which is almost exactly $7/10$:

$$\left|\sqrt{\frac{1}{2}} - \frac{7}{10}\right| < \frac{1}{100}.$$

3. Formulation of the Problem (cont-d)

- Similarly, when:
  - we start with a unit cube and
  - we want to find a cube of volume $1/2$,
  - its sides are $\sqrt[3]{1/2}$, which is almost exactly $8/10$:

$$\left|\sqrt[3]{\frac{1}{2}} - \frac{8}{10}\right| < \frac{1}{100}.$$

- So, whether we want to construct:
  - a piece of land which is (almost) exactly of half-area, or
  - a piece of gold which is (almost) exactly of half-volume,
  - decimal systems is very convenient.

4. Are There Any Other Numbers with This Property?

- Maybe here are other bases b with this property, i.e., for which, for some n_1, n_2, and n_3, we have

$$\left|\frac{1}{2}-\frac{n_1}{b}\right|<\frac{1}{b^2},\quad \left|\sqrt{\frac{1}{2}}-\frac{n_2}{b}\right|<\frac{1}{b^2},\quad \left|\sqrt[3]{\frac{1}{2}}-\frac{n_3}{b}\right|<\frac{1}{b^2}.$$

- We show that – at least for $b\le 10^9$ – only the numbers $b=2$, $b=4$, and $b=10$ satisfy this property.
- Base 4 is, in effect, the same as the binary system:
  - we group 2 binary digits (bits) to get a 4-ary digit,
  - just like we get an 8-ary system when we group 3 bits, or
  - we get a 16-based system when we group 4 bits.
- The above result may be a good explanation of why decimal and binary systems are the most widely used.

5. Considering the First Condition

- Let us first consider the first of the desired inequalities:
$$\left|\frac{1}{2}-\frac{n_1}{b}\right|<\frac{1}{b^2}.$$
- When the base is even, i.e., when $b = 2k$ for some integer k, then this property is clearly satisfied.
- Indeed, in this case, for $n_1 = k$, we get $\dfrac{n_1}{b}=\dfrac{1}{2}$ and thus, $\left|\dfrac{1}{2}-\dfrac{k}{b}\right|=0<\dfrac{1}{b^2}$.
- On the other hand:
  - if b is odd, i.e., if $b = 2k + 1$ for some natural number $k \geq 1$,
  - then, for $\dfrac{1}{2}=\dfrac{k+0.5}{2k+1}=\dfrac{k+0.5}{b}$, the closest fractions of the type $\dfrac{n_1}{b}$ are the fractions $\dfrac{k}{b}$ and $\dfrac{k+1}{b}$.

6. Considering the First Condition

- For both fractions $\frac{k}{b}$ and $\frac{k+1}{b}$, we have

$$\left|\frac{k+0.5}{2k+1}-\frac{k}{2k+1}\right|=\left|\frac{k+0.5}{2k+1}-\frac{k+1}{2k+1}\right|=$$

$$\frac{0.5}{2k+1}=\frac{1}{2\cdot(2k+1)}=\frac{1}{2b}.$$

- The desired inequality thus takes the form $\frac{1}{2b}<\frac{1}{b^2}$, which is equivalent to $2b>b^2$ and $2>b$.
- However, odd bases start with $b=3$.
- So, the 1st condition cannot be satisfied by odd bases b.
- Thus, the first condition is equivalent to requiring that the base b is an even number.

7. How Do We Check the Second Condition

- We can check the condition $\left|\sqrt{\frac{1}{2}} - \frac{n_2}{b}\right| < \frac{1}{b^2}$ literally.
- This means that we need to consider all possible values n_2 from 0 to b.
- However, this can avoided if we:
  - multiply both sides of the inequality by b, and
  - consider the equiv. inequality $\left|b \cdot \sqrt{\frac{1}{2}} - n_2\right| < \frac{1}{b}$.
- In this case, we can easily see that n_2 is the nearest integer to the product $b \cdot \sqrt{\frac{1}{2}}$: $n_2 = \left[b \cdot \sqrt{\frac{1}{2}}\right]$.
- In these terms, the desired inequality takes the form $\left|b \cdot \sqrt{\frac{1}{2}} - \left[b \cdot \sqrt{\frac{1}{2}}\right]\right| < \frac{1}{b}$; this is what we will check.

8. How to Check the Third Condition

- We can check the condition $\left|\sqrt[3]{\frac{1}{2}} - \frac{n_3}{b}\right| < \frac{1}{b^2}$ literally.
- This means that we need to consider all possible values n_3 from 0 to b.
- However, this can avoided if we:
  - multiply both sides of the inequality by b and
  - consider the equiv. inequality $\left|b \cdot \sqrt[3]{\frac{1}{2}} - n_3\right| < \frac{1}{b}$.
- In this case, we can easily see that n_3 is the nearest integer to the product $b \cdot \sqrt[3]{\frac{1}{2}}$: $n_3 = \left[b \cdot \sqrt[3]{\frac{1}{2}}\right]$.
- In these terms, the desired inequality takes the form $\left|\sqrt[3]{\frac{1}{2}} - \left[b \cdot \sqrt[3]{\frac{1}{2}}\right]\right| < \frac{1}{b}$; this is what we will check.

9. The Checking and Its Results

- For each even number b from 2 to 10^9, we checked whether this number satisfies both conditions

$$\left|b\cdot\sqrt{\frac{1}{2}}-\left[b\cdot\sqrt{\frac{1}{2}}\right]\right|<\frac{1}{b};\quad \left|\sqrt[3]{\frac{1}{2}}-\left[b\cdot\sqrt[3]{\frac{1}{2}}\right]\right|<\frac{1}{b}.$$

- *Result:* for $b\le 10^9$, both roots are only well approximated for $b=2$, $b=4$, and $b=10$.
- *Conclusion:* only for these three bases, the desired conditions are satisfied.
- This may explain why decimal and binary systems are the most frequently used.
- *What we did:* checked all the values b until 10^9.
- *Conjecture:* no other value $b>10^9$ satisfies this approximation property.

10. Java Code

```
double value;
//Loop that iterates from 2 to 1,000,000,000
for(int b = 2; b <= 1000000000; b += 2){
    value = Math.sqrt(0.5) * b;
    //Checks if the square root is well approximated
    if(Math.abs(value - Math.round(value)) < 1./b){
        value = Math.cbrt(0.5) * b;
        //Checks if the cubic root is well approximated
        if(Math.abs(value - Math.round(value)) < 1./b){
        System.out.println("Square and cubic roots "
            + "are well approximated in base " + b);
        }
    }
}
```