Attraction-Repulsion Forces Between Biological Cells: A Theoretical Explanation of Empirical Formulas

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Biological Calls Interact

Qualitative . . .

Analyticity

Tidal Forces

Scale Invariance

Definitions

Main Results

Proofs

1. Biological Calls Interact

- Biological cells attract and repulse each other.

- For each type of cell, there is a certain distance R_1 at which there is no interaction.

- When $r < R_1$, the cells repulse with the force

$$\mathbf{f} = -k_1 \cdot \left(\frac{1}{r} - \frac{1}{R_1} \right) \cdot \mathbf{e}, \text{ where } \mathbf{e} \stackrel{\text{def}}{=} \frac{\mathbf{r}}{r}.$$

- When $r > R_1$, the cells attract each other with the force $\mathbf{f} = k_2 \cdot (r - R_1) \cdot \mathbf{e}$.

- As a result of these two forces, the cells stay at the same – biologically optimal – distance from each other.

- In this paper, we provide a theoretical explanation for the above empirical formulas.

2. Qualitative Requirements: Monotonicity

- We want to find the dependence $f(r)$ of the interactive force f on the distance r between the two cells.

- To find such a dependence, let us consider natural requirements on $f(r)$.

- The larger the difference between the actual distance r and R_1, the larger should be the force.

- So, the repulsion force should increase when the distance r decreases.

- The attraction force should increase as the distance r increases.

- It should be mentioned that the empirical formulas satisfy this property.

3.   Analyticity

- All dependencies in fundamental physics are analytical functions, i.e., can be expanded in Laurent series:

$$f(r) = a_0 + a_1 \cdot r + a_2 \cdot r^2 + \ldots + a_{-1} \cdot r^{-1} + a_{-2} \cdot r^{-2} + \ldots$$

- In fundamental physics, this phenomenon is usually explained by the need to consider quantum effects:

  - quantum analysis means extension to complex numbers, and

  - analytical functions are, in effect, differential functions of complex variables.

- It is worth mentioning that both empirical formulas are analytical.

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Qualitative . . .

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Tidal Forces

Scale Invariance

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Main Results

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4. Tidal Forces

- The main objective of the forces between the two cells are to keep the cells at a certain distance.

- There is also an undesired side effect, caused by the fact that cells are not points.

- Different parts of the cell have slightly difference force acting on them.

- We have tidal forces that make the parts of the cell move with respects to each other.

- So, cells compress or stretch.

- In general, the tidal forces are proportional to the gradient of the force field $F(r) \stackrel{\text{def}}{=} \dfrac{df}{dr}$.

- From the biological viewpoint, tidal forces are undesirable, so they should be as small as possible.

5.   Scale Invariance

- Physical laws are formulated in terms of the numerical values of physical quantities.

- However, these numerical values depend on what measuring unit we select to describe this quantity.

- If we first measure distances in m, and then start using cm, then all the numerical values multiply by 100.

- In particular, 2 m becomes 200 cm.

- In most fundamental physical laws, there is no physically preferred unit.

- It thus make sense to require that the physical law not depend on the choice of the unit.

- If we change the unit of one of the quantities, then we have to change the units of related quantities.

6. Scale Invariance (cont-d)

- After an appropriate re-scaling of all the units, all the formulas should remain the same.

- Scale-invariance of the dependence $b = B(a)$ means that for every λ, there exists a $\mu(\lambda)$ such that:

  - if we change a to $a' = \lambda \cdot a$ and b to $b' = \mu(\lambda) \cdot b$,
  - the dependence remain the same: if $b = B(a)$, then we should have $b' = B(a')$, i.e., $\mu(\lambda) \cdot b = B(\lambda \cdot a)$.

- For the dependence $f(r)$, there is no scale-invariance: there is a special distance R_1 (when the force is 0).

- However, for the *tidal force* $F(r)$, both $F(r) \sim r^{-2}$ for small r and $F(r) = \text{const}$ for large r are scale invariant.

Biological Calls Interact

Qualitative . . .

Analyticity

Tidal Forces

Scale Invariance

Definitions

Main Results

Proofs

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Go Back

Full Screen

Close

Quit

7. Definitions

- By a *force function*, we mean a function $f(r)$ from positive numbers to real numbers.

- We say that a force function unction $f(r)$ is *analytical* if it can be expanded in Laurent series

$$f(r) = a_0 + a_1 \cdot r + a_2 \cdot r^2 + \ldots + a_{-1} \cdot r^{-1} + a_{-2} \cdot r^{-2} + \ldots$$

- We say that a force function is *monotonic-at-0* if for sufficiently small r, $|f(r)|$ increases as r decreases.

- We say that a force function is *monotonic-at-∞* if for sufficiently large r, $|f(r)|$ increases as r increases.

- By a *tidal force function* corresponding to the force function $f(r)$, we mean its derivative $F(r) = \dfrac{df}{dr}$.

- We say that a tidal force function is *scale-invariant* if

$$\forall \lambda > 0 \, \exists \mu(\lambda) \, \forall r \, \forall a \, (a = F(r) \Rightarrow \mu(\lambda) \cdot a = F(\lambda \cdot r)).$$

8. Definitions (cont-s)

- Let $f(r)$ be an analytical monotonic-at-0 force function $f(r)$, let $F(r)$ be its tidal force function.

- We say that $F(r)$ *grows fast* if there exists another analytical monotonic-at-0 force function $g(r)$:

  - with scale-invariant tidal force function $G(r)$,
  - for which $\dfrac{F(r)}{G(r)} \to \infty$ as $r \to 0$.

- Let $f(r)$ be an analytical monotonic-at-∞ force function $f(r)$, let $F(r)$ be its tidal force function.

- We say that $F(r)$ *grows fast* if there exists another analytical monotonic-at-∞ force function $g(r)$,

  - with scale-invariant tidal force function $G(r)$,
  - for which $\dfrac{F(r)}{G(r)} \to \infty$ as $r \to 0$.

9. Main Results

- **Proposition 1.** *Every analytical monotonic-at-0 force function $f(r)$*

  - *for which the tidal force function $F(r)$ is scale-invariant and does not grow fast,*
  - *has the form $f(r) = \dfrac{c_0}{r} + c_1$ for some c_0 and c_1.*

- **Proposition 2.** *Every analytical monotonic-at-∞ force function $f(r)$*

  - *for which the tidal force function $F(r)$ is scale-invariant and does not grow fast,*
  - *has the form $f(r) = c_0 \cdot r + c_1$ for some c_0 and c_1.*

- These are exactly the empirical formulas that we wanted to explain.

- Thus, we have a theoretical explanation for these formulas.

Biological Calls Interact

Qualitative . . .

Analyticity

Tidal Forces

Scale Invariance

Definitions

Main Results

Proofs

10. Proofs

- The tidal force function $F(r)$ is scale-invariant:

$$F(\lambda \cdot r) = \mu(\lambda) \cdot F(r). \qquad (1)$$

- The function $F(r)$ is analytical, thus smooth.

- Thus, $\mu(\lambda) = \dfrac{F(\lambda \cdot r)}{F(r)}$ is smooth as the ratio of two smooth functions.

- Differentiating both sides of (1) by λ and taking $\lambda = 1$, we get $r \cdot \dfrac{dF}{dr} = \alpha \cdot F$, where $\alpha \stackrel{\text{def}}{=} \dfrac{d\mu}{d\lambda}_{|\lambda=1}$.

- Moving all terms with r to one side and all terms with F to another, we get $\dfrac{dF}{F} = \alpha \cdot \dfrac{dr}{r}$.

- Integrating both sides, we get $\ln(F(r)) = \alpha \cdot \ln(r) + C$, for some integration constant C.

11. Proofs (cont-d)

- Thus, for $F(r) = \exp(\ln(F(r)))$, we get $F(r) = c \cdot r^\alpha$, where $c \stackrel{\text{def}}{=} \exp(C)$.

- Since $f(r)$ is analytical, its derivative $F(r)$ is also analytical; thus, α is an integer.

- For $\alpha = -1$, integration of $F(r)$ leads to not-analytical function $f(r) = c \cdot \ln(r)$.

- Thus, $\alpha \neq -1$, and integration of $F(r)$ leads to $f(r) = c_0 \cdot r^{\alpha+1} + c_1$, where $c_0 \stackrel{\text{def}}{=} \dfrac{c}{\alpha + 1}$.

- Monotonicity-at-0 implies that $\alpha + 1 < 0$, i.e., that that $\alpha + 1 \leq -1$ and $\alpha \leq -2$.

- For $\alpha < -2$, we could take $g(r) = r^{-1}$ with $G(r) = -r^{-2}$ and thus, $\dfrac{F(r)}{G(r)} \sim \dfrac{r^\alpha}{r^{-2}} = r^{\alpha+2}$.

Biological Calls Interact

Qualitative . . .

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Tidal Forces

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Definitions

Main Results

Proofs

12. Proofs (final)

- From $\alpha < -2$, it follows that $\alpha + 2 < 0$, hence $\dfrac{F(r)}{G(r)} \sim r^{\alpha+2} \to \infty$ as $r \to 0$.

- So, if $\alpha < -2$, the tidal force function grows fast.

- The only case when this function does not grow fast is the case $\alpha = -2$, which leads to $f(r) = c_0 \cdot r^{-1} + c_1$.

- Similarly, monotonicity-at-∞ implies that $\alpha + 1 > 0$, i.e., that that $\alpha + 1 \geq 1$ and $\alpha \geq 0$.

- For $\alpha > 0$, we could take $g(r) = r$ with $G(r) = 1$ and thus, $\dfrac{F(r)}{G(r)} \sim r^{\alpha}$.

- From $\alpha > 0$, $r^{\alpha} \to \infty$ as $r \to \infty$.

- So, is $\alpha > 0$, the tidal force function grows fast.

- The only case when this function does not grow fast is the case $\alpha = 0$, which leads to $f(r) = c_0 \cdot r + c_1$. Q.E.D.

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Home Page

Title Page

◀◀ ▶▶

◀ ▶

Page 14 of 14

Go Back

Full Screen

Close

Quit