Reverse Mathematics Is Computable for Interval Computations

Martine Ceberio[1], Olga Kosheleva[2], and Vladik Kreinovich[1]

Departments of [1]Computer Science and [2]Teacher Education
University of Texas at El Paso
500 W. University, El Paso, TX 79968, USA
mceberio@utep.edu, olgak@utep.edu,
vladik@utep.edu

1. What Is Reverse Mathematics

- In mathematics:
  - whenever a new theorem is proven, often,
  - it later turns out that this same conclusion can be proven under weaker conditions.
- For example, first, it was proven that
  - if for a continuous function $f(x)$ from real numbers to real numbers, we have
    $$\forall a\,\forall b\,(f(a+b)=f(a)+f(b)),$$
  - then this function $f(x)$ is linear, i.e., $f(a)=k\cdot a$ for some k,
- Later on, it turned out that the same is true:
  - not only for continuous functions,
  - but also for all measurable functions.

2. What Is Reverse Mathematics (cont-d)

- Because of this phenomenon:
  - every time a new result is proven,
  - researchers start analyzing whether this result can be proven under weaker conditions.
- In the past, usually, weaker and weaker conditions were found.
- Lately, however, in some problems, it has become possible to find
  - the weakest possible conditions
  - under which the given conclusion is true.

3. What Is Reverse Mathematics (cont-d)

- From the logical viewpoint,
  - the fact that the condition A in the implication $A \Rightarrow B$ cannot be weakened
  - means that A is equivalent to B, i.e., that also
  $$B \Rightarrow A;$$
  - in other words, that we can *reverse* the implication.
- Because of this, the search for such weakest possible condition is known as the *reverse mathematics.*

4. What Does Reverse Mathematics Mean for Interval Computations?

- The main problem of interval computations is:
  - given an algorithm $f(x_1, \ldots, x_n)$ and ranges $[\underline{x}_i, \overline{x}_i]$,
  - to find the range $[\underline{y}, \overline{y}]$ of possible values of $y = f(x_1, \ldots, x_n)$ when $x_i \in [\underline{x}_i, \overline{x}_i]$ for all i.
- In practice, this can be used, e.g., to check that:
  - under all possible values of the parameters x_i from the corresponding intervals,
  - the system is stable.
- Indeed, stability can be described by an inequality $f(x_1, \ldots, x_n) \le y_0$ for some value y_0.
- Once we know the range, this checking is equivalent to simply checking whether $\overline{y} \le y_0$.

5. Reverse Mathematics for Interval Computations (cont-d)

- Similarly, if $f(x_1, \ldots, x_n)$ is the amount of polluting chemical released by a plant under conditions x_i, then:
  - checking whether the level of this chemical never exceeds the desired threshold y_0
  - is also equivalent to checking whether $\overline{y} \le y_0$.
- In addition to knowing that $x_i \in [\underline{x}_i, \overline{x}_i]$, we often have additional constraints on the values x_i.
- This makes the problem more complex.
- We may also need to check more complex conditions.

6. Reverse Mathematics for Interval Computations (cont-d)

- For example, in solving system of equations under interval uncertainty, we are often interested in
  - finding all the values $x = (x_1, \ldots, x_n)$ for which,
  - for all possible values $a_1, \ldots, a_m$ from the corresponding intervals,
  - there exist appropriate controls $c_1, \ldots, c_p$ from the given intervals for which $f(x, a, c) \le y_0$.
- All physical quantities are bounded.
- So, we can safely assume that all variables in the quantifiers are bounded.
- In general, we have a property $P(x_1, \ldots, x_n)$.
- This can be a simple inequality like $f(x_1, \ldots, x_n) \le y_0$.

7. Reverse Mathematics for Interval Computations (cont-d)

- This can also be a complex formula obtained from simply inequalities by using:
  - logical connectives “and” (&) “or” ($\vee$), and “not” ($\neg$), and
  - quantifiers $\forall x$ and $\exists x$ over real numbers.
- By using interval methods, we find a box B for which the desired property $P(x)$ holds for all points $x \in B$:

$$B = [\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_n, \overline{x}_n].$$

- In this context, reverse mathematics means trying to find:
  - not just this box,
  - but also the whole set of all the tuples x for which the property $P(x)$ holds.

8. What We Do in This Talk

- In the interval context, reverse mathematics means trying to find:
  - not just this box,
  - but also the whole set of all the tuples x for which the property $P(x)$ holds.
- In this talk, we show that such a set can be indeed computed:
  - maybe not exactly,
  - but at least with any possible accuracy.

9. Definitions

- According to computable mathematics, a real number x is *computable* if there exists an algorithm that,
  - given a natural number n,
  - generates a rational number r_n for which

$$|r_n - x| \leq 2^{-n}.$$

- A tuple of computable number is called a *computable tuple.*
- A bounded set S is called *computable* if there exists an algorithm, that:
  - given a natural number n,
  - generates a finite list S_n of computable tuples for which $d_H(S, S_n) \leq 2^{-n}$.

10. Definitions (cont-d)

- Here, $d_H(A, B)$ is the Hausdorff distance: the smallest $\varepsilon > 0$ for which:
  - every element $a \in A$ is ε-close to some element $b \in B$, and
  - every element $b \in B$ is ε-close to some element $a \in A$.
- A *computable function* is a function $f(x_1, \ldots, x_n)$ for which the following two algorithms exist.
- The main algorithm that:
  - given rational values $r_1, \ldots, r_n$,
  - returns a computable number $f(r_1, \ldots, r_n)$.

11. Definitions and Results

- An auxiliary algorithm that:
  - given a rational number $\varepsilon > 0$,
  - returns a rational number $\delta > 0$ for which $d(x, x') \le \delta$ implies $d(f(x), f(x')) \le \varepsilon$.
- Most arithmetic and elementary functions are everywhere computable.
- The only exceptions are discontinuous functions like sign or tangent.
- It is known (and it can be easily proven):
  - that min and max are computable,
  - that composition of two computable functions is computable, and
  - that the maximum and minimum of a computable function over a computable set are also computable.

12. Definitions and Results (cont-d)

- It is also known that:
  - for every computable function f on a computable set S, and
  - for every two values $y^- < y^+$ for which

$$\min_{x \in S} f(x) < y^-,$$

  - there exists a value $y_0 \in [y^-, y^+]$ for which the set $\{x : f(x) \le y_0\}$ is computable.
- There are also known negative results.
- It is not possible, given two computable numbers x and x', to check whether $x \le x'$.
- As a consequence, it is, in general, not possible:
  - given a computable function f and a number y,
  - to produce a computable set $\{x : f(x) \le y_0\}$.

13. Definitions and Results (cont-d)

- Otherwise, for a constant $f(x) = c$, we would get an algorithm for checking whether $c \le y_0$.
- Let $v_1, \ldots$ be real-valued variables.
- For each of these variables, we have bounds

$$\underline{V}_i \le v_i \le \overline{V}_i.$$

- By a *term*, we mean an expression of the type $f(v_{i_1}, \ldots, v_{i_m})$, where:
  - f is a computable function and
  - v_i are given variables.
- By an *elementary formula*, we means an expression $t_1 < t_2$, $t_1 \le t_2$, or $t_1 = t_2$, where t_i are terms.

14. Definitions and Results (cont-d)

- By a *property* $P(x_1, \ldots, x_n)$, we mean any formula with free variables $x_1, \ldots, x_n$ which is:
  - obtained from elementary formulas
  - by using logical connectives &, $\vee$, $\neg$, and quantifiers $\forall v_{i_{\in[\underline{V}_i,\overline{V}_i]}}$ and $\exists v_{i_{\in[\underline{V}_i,\overline{V}_i]}}$.
- To simplify, we will represent each equality $t_1 = t_2$ as two inequalities $t_1 \le t_2$ and $t_2 \le t_1$.
- Let $\varepsilon > 0$ be a real number.
- We say that elementary formulas $t_1 \le t_2$ (or $t_1 < t_2$) and $t_1 \le t_2 + \varepsilon'$ (or $t_1 < t_2 + \varepsilon'$) are ε-*close* if $|\varepsilon'| \le \varepsilon$.

15. Definitions and Results (cont-d)

- We say that the formulas $P(x_1, \ldots, x_n)$ and $P(x'_1, \ldots, x'_n)$ are ε-*close* if:
  - the formula P' is the result of replacing, in the formula P,
  - each elementary formula with an ε-close one.
- In practice, all the values are measured with some accuracy.
- Thus, if ε is sufficiently small, the two ε-close elementary formulas are practically indistinguishable.
- Thus, in general, ε-close properties are indistinguishable as well.

16. Main Results

- **Proposition.**
  - *Let $P(x_1, \ldots, x_n)$ be a property which is satisfied for all the tuples x from a given box.*
  - *Then, we can compute the set $\{x : P'(x)\}$ for some property P' which is ε-close to P.*
- This result can be further strengthened.
- **Proposition.** *Let $P(x_1, \ldots, x_n)$ be satisfied for all x from B; then, we can compute $S = \{x : P'(x)\}$, where:*
  - *the property P' is ε-close to P and*
  - *we have $S'' \stackrel{\text{def}}{=} \{x : P''(x)\} \subseteq S$ for all properties P'' which are $(\varepsilon/2)$-close to P.*

17. Proof of Proposition 1

- First, let us transform the original property $P(x)$ into a prenex normal form, i.e., into the form in which:
  - we first have quantities, and
  - then the quantifier-free part.
- Indeed, if we have a logical connective outside quantifires, we can move the quantifier out:

$$\neg\forall x P \to \exists x \neg P, \quad \forall x P \vee Q \to \forall x (P \vee Q),$$

$$\forall x P \,\&\, Q \to \forall x (P \,\&\, Q), \quad \neg\exists x P \to \forall x \neg P,$$

$$\exists x P \vee Q \to \exists x (P \vee Q), \quad \exists x P \,\&\, Q \to \exists x (P \,\&\, Q).$$

- Then, we can use de Morgan rules to move all negations inside:

$$\neg(A \,\&\, B) \to (\neg A) \vee (\neg B) \text{ and } \neg(A \vee B) \to (\neg A) \,\&\, (\neg B).$$

18. Proof of Proposition 1 (cont-d)

- When applied to an elementary formulas $t_1 \le t_2$ or $t_1 < t_2$, negation simply means:

$$\neg(t_1 \le t_2) \to t_2 < t_1 \text{ and } \neg(t_1 < t_2) \to t_2 \le t_1.$$

- In the resulting formula, let us replace all $<$ with $\le$; this will not change ε-closeness.
- Let us denote the result by P_0.
- Let us describe the P_0 in the equivalent form $F(x_1, \ldots, x_n) \le 0$, for some computable function F.
- Each elementary formula $t_1 \le t_2$ can be equivalently reformulated as $t_1 - t_2 \le 0$.
- Each formula $(F_1 \le 0) \vee (F_2 \le 0)$ can be equivalently reformulated as $\min(F_1, F_2) \le 0$.
- Each formula $(F_1 \le 0) \,\&\, (F_2 \le 0)$ can be equivalently reformulated as $\max(F_1, F_2) \le 0$.

19. Proof of Proposition 1 (cont-d)

- Each formula $\exists v_{i_{\in[\underline{V}_i,\overline{V}_i]}} F(v_i,\ldots) \le 0$ can be equivalently reformulated as $\min\limits_{v_i\in[\underline{V}_i,\overline{V}_i]} F(v_i,\ldots) \le 0$.
- Each formula $\forall v_{i_{\in[\underline{V}_i,\overline{V}_i]}} F(v_i,\ldots) \le 0$ can be equivalently reformulated as $\max\limits_{v_i\in[\underline{V}_i,\overline{V}_i]} F(v_i,\ldots) \le 0$.
- For the resulting function $F(x_1,\ldots,x_n)$, for $y^- = 0$ and $y^+ = \varepsilon$:
  - there exists a number $\varepsilon_0 \in (0,\varepsilon)$
  - for which the set $S_0 \stackrel{\text{def}}{=} \{x : F(x) \le \varepsilon_0\}$ is computable.
- The corresponding inequality $F(x) \le \varepsilon_0$ is equivalent to $F'(x) \le 0$, where $F'(x) \stackrel{\text{def}}{=} F(x) - \varepsilon_0$.

20. Proof of Proposition 1 (cont-d)

- This inequality can be obtained if we replace, in P_0:
  - each elementary formula $t_1 \le t_2$
  - with a formula $t_1 \le t_2 + \varepsilon_0$.
- Since $\varepsilon_0 < \varepsilon$, this transformation keeps all elementary formulas ε-close to the original ones.
- So, the resulting formula P'_0 is ε-close to the formula P_0 and we have $S_0 = \{x : P'_0(x)\}$.
- When we went from P to P_0, all we did was changed the sign of some inequalities.
- This, in turn, can be obtained by appropriately changing the elementary formulas from P to ε-close ones.
- Thus, indeed, the set S_0 can be represented as $S_0 = \{x : P'(x)\}$, where P' is ε-close to P.

21. Proof of Proposition 2

- This proof is similar, except that instead of $y^- = 0$ we take $y^- = \varepsilon/2$.
- Then, for every property P'' which is $(\varepsilon/2)$-close to P:
  - on each level of designing a function $F(x_1, \ldots, x_n)$,
  - we will have $F \le F'' + \varepsilon/2$ for the function F'' corresponding to the property P''.
- Thus, at the end, we conclude that $F \le F'' + \varepsilon/2$.
- Since now $\varepsilon/2 < \varepsilon_0$, we conclude that $F(x) \le F''(x) + \varepsilon_0$ and $F'(x) = F(x) - \varepsilon_0 \le F''(x)$.
- Thus, $F''(x) \le 0$ implies that $F'(x) \le 0$. So, indeed, $P'' \subseteq S'$.
- The proposition is proven.

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