Derivation of Louisville-Bratu-Gelfand Equation from Shift- or Scale-Invariance

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1. In Many Different Situations, We Have the Exact Same Louisville-Bratu-Gelfand Equation

- In many different physical situations, we encounter the same differential equation

$$\nabla^2 \varphi = c \cdot \exp(a \cdot \varphi).$$

- This equation – known as Louisville-Bratu-Gelfand equation – appears:
  - in the analysis of explosions,
  - in the study of combustion,
  - in astrophysics (to describe the matter distribution in a nebula),
  - in electrodynamics – to describe the electric space charge around a glowing wire – and
  - in many other applications areas.

2. Challenge

- The same equation appears in many different situations.
- This seems to indicate that this equation should not depend on any specific physical process.
- It should be possible to derive it from general principles.
- In this paper, we show that this equation can be naturally derived from basic symmetry requirements.

3. Laplace Equation

- The simplest form of our equation is when $c = 0$.
- In this case, we get a linear equation $\nabla^2 \varphi = 0$.
- This equation is known as the Laplace equation; so:
  - in order to understand where our equation comes from,
  - let us first recall where the Laplace equation comes from.

4. Scalar Fields Are Ubiquitous

- To describe the state of the world, we need to describe the values of all physical quantities at all locations.
- In physics, the dependence $\varphi(x)$ of a physical quantity φ on the location x is known as a *field*.
- Typical examples are components of an electric or magnetic fields, gravity field, etc.
- In general, at each location x, there are many different physical fields.
- In some cases, several fields are strong enough to affect the situation.
- So, we need to take several fields into account.
- However, in many practical situations, only one field is strong enough.

5. Scalar Fields Are Ubiquitous (cont-d)

- For example:
  - when we analyze the motion of celestial bodies,
  - we can safely ignore all the fields except for gravity.
- Similarly, if we analyze electric circuits, we can safely ignore all the fields but the electromagnetic field.

6. Case of Weak Fields

- In general, equations describing fields are non-linear.
- However, in many real-life situations, fields are weak.
- In this case, we can safely ignore quadratic and higher order terms in terms of φ and consider linear equations.

7. General Case of Linear Equations

- In physics, usually, we consider second order differential equations, i.e., equations that depend:
  - on the field φ,
  - on its first order partial derivatives $\varphi_{,i} \stackrel{\text{def}}{=} \dfrac{\partial \varphi}{\partial x_i}$ and
  - on its second order derivatives $\varphi_{,ij} \stackrel{\text{def}}{=} \dfrac{\partial^2 \varphi}{\partial x_i \partial x_j}$.
- The general linear equation containing these terms has the form

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} \cdot \varphi_{,ij} + \sum_{i=1}^{3} a_i \cdot \varphi_{,i} + a \cdot \varphi = 0.$$

8. Rotation-Invariance

- In general, physics does not change if we simply rotate the coordinate system.
- Thus, it is reasonable to require that the system be invariant with respect to arbitrary rotations.
- This requirement eliminates the terms proportional to the first derivatives $\varphi_{,i}$.
- Otherwise, we have a selected vector a_i and thus, an expression which is not rotation-invariant.
- Similarly, we cannot have different eigenvector of the matrix a_{ij} – this would violate rotation-invariance.
- Thus, this matrix must be proportional to the unit matrix with components $\delta_{ij} = 1$ if $i = j$ else 0.

9. Rotation-Invariance (cont-d)

- So, $a_{ij} = a_0 \cdot \delta_{ij}$ for some a_0, and the above equation takes the form $a_0 \cdot \sum\limits_{i=1}^{3} \varphi_{,ii} + a \cdot \varphi = 0$.
- Dividing both sides by a_0 and taking into account that $\sum\limits_{i=1}^{3} \varphi_{,ii} = \nabla^2 \varphi$, we get the equation

$$\nabla^2 \varphi + m \cdot \varphi = 0, \text{ where } m \stackrel{\text{def}}{=} a/a_0.$$

- This equation is indeed the general physics equation for a weak scalar field.
- The case of $m = 0$ corresponds to electromagnetic field or gravitational field.

10. Rotation-Invariance (cont-d)

- More generally, $m = 0$ corresponds to any field whose quanta have zero rest mass.
- Example: photons or gravitons, quanta of the above fields.
- In the general case, when the quanta have non-zero rest mass, we get a more general equation with $m \neq 0$.
- Example: strong interactions whose quanta are π-mesons.

11. Additional Conditions Are Needed to Pinpoint Laplace Equation.

- Can we explain why:
  - out of all possible equations of type,
  - Laplace equation – corresponding to $m = 0$ – is the most frequent?
- We need to use additional conditions.
- As such conditions, we will use the fundamental notions of scale- and shift-invariance.

12. Scale-Invariance: General Idea

- Equations deal with numbers.
- To describe the value of a physical quantity as a number, we need to select a measuring unit.
- If we change the original unit to a one which is λ times smaller, then:
  - the same physical quantity which was previously described by the number x
  - will now be described by a λ times larger number

$$x' = \lambda \cdot x.$$

- For example:
  - if we replace meters with a 100 times smaller unit – centimeter,
  - all the length values are multiplied by 100: 1.7 meters becomes $1.7 \cdot 100 = 170$ centimeters.

13. Scale-Invariance (cont-d)

- The choice of a measuring unit is rather arbitrary.
- It is therefore reasonable to require that the fundamental physical equations should not change
  - if we simply change a measuring unit.
  - i.e., if we replace x with $x' = \lambda \cdot x$.
- Of course, different quantities may be related.
- So, if we change the unit of one quantity, we may need to appropriate change units for related quantities.
- For example, if we change the unit of time t, e.g., for hours to seconds, then:
  - to preserve the relation $d = v \cdot t$ between the velocity v and the distance d,
  - we need to also change the unit for measuring velocity – e.g., from km/hour to km/sec.

14. Two Quantities

- Our equation involves two physical quantities: the physical field φ and the coordinate (distance) x_i.
- Thus, we can consider scale-invariance with respect to both these quantities.
- The equation is linear in φ.
- So, it does not change if we replace the original field $\varphi(x)$ with a φ-re-scaled field $\varphi'(x) = \lambda \cdot \varphi(x)$.
- If we change the unit of measuring x_i to a unit which is λ times smaller, then the numerical values will change:
$$x_i \to \lambda \cdot x_i.$$
- Thus, each derivative $\dfrac{\partial}{\partial x_i}$ gets divided by λ, and so, the second derivative is divided by λ^2.
- The term $m \cdot \varphi$ remains unchanged.

15. Two Quantities (cont-d)

- As a result, the above equation changes into

$$\frac{1}{\lambda^2}\cdot\nabla^2\varphi + m\cdot\varphi = 0.$$

- This is equivalent to $\nabla^2\varphi + m\cdot\lambda^2\cdot\varphi = 0.$
- The only case when this equation is equivalent to the original one is when the coefficients at φ are equal:

$$m = m\cdot\lambda^2 \text{ thus } m = 0.$$

- In other words, the only x-scale-invariant case of the general linear equation is the Laplace equation.

16. Shift-Invariance: General Idea

- For many physical quantities, the numerical value also depends on the selection of the starting point.
- Examples: time, coordinate.
- If we change the starting point of measuring time to a new one which is s moments before, then:
  - instead of the original measurement results t,
  - we will get new shifted numerical values $t' = t + s$.
- The selection of a starting point is simply a matter of convenience, there is nothing fundamental about it.
- It is therefore reasonable to require that:
  - the fundamental physical equations do not change
  - if we simply change the starting point.

17. Shift-Invariance (cont-d)

- Of course:
  - to preserve the equations,
  - we may need to accordingly change measuring unit or a starting point) for some other quantities.
- Let us see what we can conclude in our case by requiring shift-invariance for x_i and for φ.

18. x- and φ-Shift-Invariance

- Let us replace the original variables x_i with new variables $x'_i = x_i + s_i$, where s_i is the shift.
- Then the derivatives do not change and thus, the equation remains the same.
- Let us now consider the consequences of requiring that the equation are invariant with respect to shifting φ:

$$\varphi'(x) = \varphi(x) + s.$$

- In many cases, such a shift makes perfect physical sense.
- Indeed, e.g.:
  - the only way we measure electric potential $\varphi(x)$
  - is by measuring the difference $\varphi(x) - \varphi(x')$ between potentials at different locations.

19. x- and φ-Shift-Invariance (cont-d)

- If we add the same value s to all the values of the field:
  - then the differences remain the same,
  - thus, measurement results.
- What happens if we apply this shift to our equation?
- The derivatives do not change (since the derivative of a constant s is 0).
- The term $m \cdot \varphi$ changes into $m \cdot (\varphi + s)$.
- Thus, instead of the original equation, we get a new equation $\nabla^2 \varphi + m \cdot \varphi + m \cdot s = 0$.
- The resulting equation is equivalent to the original when $m \cdot s = 0$, i.e., when

$$m = 0.$$

- In other words, the only φ-shift-invariant case of the general linear equation is the Laplace equation.

20. Summary

- To get Laplace equation $\nabla^2\varphi = 0$ out of the general linear equation, we need to postulate:
  - either x-scale-invariance
  - or φ-shift-invariance.

21. From Laplace Equation to Poisson Equation

- The Laplace equation $\nabla^2 \varphi = 0$ describes what happens in the absence of any external sources.
- If there is an external source, then the expression $\nabla^2 \varphi$ is, in general, not necessarily equal to 0.
- In other words, we have an equation $\nabla^2 \varphi = f$ for some external function f.
- This equation is known as the Poisson equation.

22. How to Describe Non-Linearity

- Non-linearity also means that the original linear equation is no longer exactly true.
- There are additional nonlinear terms in this equation.
- We can view these non-linear terms as a source for the field.
- So, in effect, we have the Poisson equation.
- The only difference is that now, the source term f is not an external term.
- It is a nonlinear function of the field itself:

$$\nabla^2 \varphi = f(\varphi).$$

- The question is: which function $f(\varphi)$ should we choose?

23. Which Function $f(\varphi)$ Should We Choose? Let Us Use Symmetries

- Let us use the same natural symmetries that we used to derive the Laplace equation in the first place:
  - φ-shift-invariance and
  - x-scale-invariance.
- When is the resulting equation φ-shift-invariant?
- If we replace the original values of the field $\varphi(x)$ with the shifted values $\varphi'(x) = \varphi(x) + s$, then:
  - the derivatives will not change,
  - so our equation will take the form $\nabla^2\varphi = f(\varphi + s)$.
- Literally speaking, these two equations coincide if for all φ and s, we have $f(\varphi + s) = f(\varphi)$.
- In this case, as we can easily see, the function f is simply a constant – so there is no nonlinearity.

24. Which $f(\varphi)$ Should We Choose (cont-d)

- Sometimes, to preserve the equation, we need to accordingly make changes with other variables as well.
- In our case, this means that:
  - in addition to a shift $\varphi \to \varphi + s$,
  - we may also need to apply an appropriate re-scaling of the coordinates x_i: $x_i \to \lambda(s) \cdot x_i$.
- Under this re-scaling, the second derivatives are divided by λ^2.
- So, we get a more complicated equation

$$\frac{1}{\lambda^2(s)} \cdot \nabla^2 \varphi = f(\varphi + s).$$

- This is equivalent to $\nabla^2 \varphi = \lambda^2(s) \cdot f(\varphi + s)$.
- For the new equation to be equivalent to the original equation, their right-hand sides must coincide.

25. Which $f(\varphi)$ Should We Choose (cont-d)

- So, for all φ and s, we have $\lambda^2(s) \cdot f(\varphi + s) = f(\varphi)$, or, equivalently,

$$f(\varphi + s) = C(s) \cdot f(\varphi), \text{ where } C(s) \stackrel{\text{def}}{=} \frac{1}{\lambda^2(s)}.$$

- In physics, all dependencies are measurable, so the function $f(\varphi)$ is measurable.
- Thus, the function $C(s) = f(\varphi + s)/f(\varphi)$ is also measurable, as the ratio of two measurable functions.
- It is known that for measurable functions, the only solutions to the above functional equation are functions

$$f(\varphi) = c \cdot \exp(a \cdot \varphi).$$

- This is exactly Louisville-Bratu-Gelfand equation that we are trying to explain!

26. What If We Require x-Scale-Invariance?

- If we replace the original values of the coordinates x_i with the re-scaled values $x'_i = \lambda \cdot x_i$, then:
  - the derivatives will divide by λ^2, while
  - the term $f(\varphi)$ will not change.
- So, our equation will take the form
$$\frac{1}{\lambda^2} \cdot \nabla^2 \varphi = f(\varphi).$$
- This is equivalent to $\nabla^2 \varphi = \lambda^2 \cdot f(\varphi)$.
- Literally speaking, these two equations coincide if $f(\varphi) = \lambda^2 \cdot f(\varphi)$ for all φ and λ.
- In this case, as we can easily see, the function f is simply 0 – so there is no nonlinearity.
- Sometimes, to preserve the equation, we need to accordingly make changes with other variables as well.

27. x-Scale-Invariance (cont-d)

- In our case, this means that we may also need to apply an appropriate shift $s(\lambda)$ of the φ-field:

$$\varphi(x) \to \varphi(x) + s(\lambda).$$

- Under this shift, the derivatives do not change, but the value $f(\varphi)$ is replaced by the value $f(\varphi + s(\lambda))$.
- So, we get a more complicated equation

$$\frac{1}{\lambda^2} \cdot \nabla^2 \varphi = f(\varphi + s(\lambda)).$$

- This is equivalent to $\nabla^2 \varphi = \lambda^2 \cdot f(\varphi + s(\lambda))$.
- For the new equation to be equivalent to the original equation, their right-hand sides must coincide.

28. x-Scale-Invariance (cont-d)

- So, for all φ and λ, we have

$$\lambda^2 \cdot f(\varphi + s(\lambda)) = f(\varphi).$$

- This is equivalent to

$$f(\varphi + s(\lambda)) = \lambda^{-2} \cdot f(\varphi).$$

- For this equation, we also get $f(\varphi) = c \cdot \exp(a \cdot \varphi)$, i.e., we also get the Louisville-Bratu-Gelfand equation.

29. General Conclusion

- The simplest case of the Louisville-Bratu-Gelfand equation is the Laplace equation $\nabla^2\varphi = 0$.
- To derive this equation from the general linear equation, we need to require:
  - either φ-shift-invariance
  - or x-scale-invariance.
- It turns out that in the nonlinear case:
  - each of these two invariance requirements
  - uniquely determines the Louisville-Bratu-Gelfand equation.
- So, this equation can be derived from natural symmetries.
- This explains why this same equation emerges in the description of many different physical phenomena.

30. Acknowledgement

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