

Natural Invariance Explains Empirical Success of Specific Membership Functions, Hedge Operations, and Negation Operations

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1. Fuzzy Techniques: A Brief Reminder

- In many applications, we have knowledge formulated:
 - in terms of imprecise (“fuzzy”) terms from natural language,
 - like “small”, “somewhat small”, etc.
- To translate this knowledge into computer-understandable form, Lotfi Zadeh proposes *fuzzy techniques*.
- According to these techniques, each imprecise property like “small” can be described by assigning:
 - to each value x of the corresponding quantity,
 - a degree $\mu(x)$ to which, according to the expert, this property is true.

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2. Fuzzy Techniques (cont-d)

- These degrees are usually selected from the interval $[0, 1]$, so that:
 - 1 corresponds to full confidence,
 - 0 to complete lack of confidence, and
 - values between 0 and 1 describe intermediate degrees of confidence.
- The resulting function $\mu(x)$ is known as a *membership function*.
- In practice, we can only ask finitely many questions to the expert.
- So we only elicit a few values $\mu(x_1)$, $\mu(x_2)$, etc.
- Based on these values, we need to estimate the values $\mu(x)$ for all other values x .

3. Fuzzy Techniques (cont-d)

- For this purpose, usually:
 - we select a family of membership functions – e.g., triangular, trapezoidal, etc. – and
 - we select a function from this family which best fits the known values.
- For terms like “somewhat small”, “very small”, the situation is more complicated.
- We can add different “hedges” like “somewhat”, “very”, etc., to each property.
- As a result, we get a large number of possible terms.

4. Fuzzy Techniques (cont-d)

- It is not realistically possible to ask the expert about each such term; instead:
 - practitioners estimate the degree to which, e.g., “somewhat small” is true
 - based on the degree to which “small” is true.
- In other words, with each linguistic hedge, we associate a function h from $[0, 1]$ to $[0, 1]$ that:
 - transforms the degree to which a property is true
 - into an estimate for the degree to which the hedged property is true.

5. Fuzzy Techniques (cont-d)

- Similarly to the membership functions:
 - we can elicit a few values $h(x_i)$ of the hedge operation from the experts, and
 - then we extrapolate and/or interpolate to get all the other values of $h(x)$.
- Usually, a family of hedge operations is pre-selected.
- Then we select a specific operation from this family which best fits the elicited values $h(x_i)$.

6. Fuzzy Techniques (cont-d)

- Similarly:
 - instead of asking experts for their degrees of confidence in statements like “not small”,
 - we estimate these degrees based on their degrees of confidence in the positive statements.
- The corresponding operation $n(x)$ is known as the *negation operation*.

7. Need to Select Proper Membership Functions, Hedge Operations, And Negation Operations

- Fuzzy techniques have been successfully applied to many application areas.
- However, this does not necessarily mean that every time we try to use fuzzy techniques, we get a success.
- The success (or not) often depends on which membership functions etc. we select:
 - for some selections, we get good results (e.g., good control),
 - for other selections, the results are not so good.
- There is a lot of empirical data about which selections work better.
- In this talk, we provide a general explanation for several of these empirically best selections.

8. Need to Select Proper Functions (cont-d)

- This explanation is based on the natural concepts of invariance.
- For symmetric membership functions that describe properties like “small”,
 - for which $\mu(x) = \mu(-x)$ and the degree $\mu(|x|)$ decreases with $|x|$,
 - in many practical situations, the most empirically successful are so-called *distending* functions:

$$\mu(x) = \frac{1}{1 + a \cdot |x|^b}.$$

- Among hedge and negation operations, often, the most efficient are fractional linear functions:

$$h(x) = \frac{a + b \cdot x}{1 + c \cdot x}.$$

9. Re-Scaling

- The variable x describes the value of some physical quantity, such a distance, height, etc.
- When we process these values, we deal with numbers.
- Numbers depend on the selection of the measuring unit:
 - if we replace the original measuring unit with a new one which is λ times smaller,
 - then all the numerical values will be multiplied by λ : $x \rightarrow X = \lambda \cdot x$.
- For example, 2 meters become $2 \cdot 100 = 200$ cm.
- This transformation from one measuring scale to another is known as *re-scaling*.

10. Scale-Invariance: Idea

- In many physical situations, the choice of a measuring unit is rather arbitrary.
- In such situations, all the formulas remain the same no matter what unit we use.
- For example, the formula $y = x^2$ for the area of the square with side x remains valid:
 - if we replace the unit for measuring sides from meters with centimeters,
 - of course, we then need to appropriately change the unit for y , from m^2 to cm^2 .

11. Scale-Invariance (cont-d)

- In general, invariance of the formula $y = f(x)$ means that:
 - for each re-scaling $x \rightarrow X = \lambda \cdot x$, there exists an appropriate re-scaling $y \rightarrow Y$
 - for which the same formula $Y = f(X)$ will be true for the re-scaled variables X and Y .

12. Let Us Apply This Idea to the Membership Function

- It is reasonable to require that:
 - the selection of the best membership functions
 - should also not depend on the choice of the unit for measuring the corresponding quantity x .
- So, it is reasonable to require that for each $\lambda > 0$:
 - there should exist some reasonable transformation $y \rightarrow Y = T(y)$ of the degree of confidence
 - for which $y = \mu(x)$ implies $Y = \mu(X)$.

13. So, What Are Reasonable Transformations of the Degree of Confidence?

- One way to measure the degree of confidence is to have a poll:
 - ask N experts how many of them believe that a given value x is, e.g., small,
 - count the number M of those who believe in this, and
 - take the ratio M/N as the desired degree $y = \mu(x)$.
- As usual with polls, the more people we ask, the more adequately we describe the general opinion.
- So, to get a more accurate estimate for $\mu(x)$, it is reasonable to ask more people.
- When we have a limited number of people to ask, it is reasonable to ask top experts in the field.

14. Reasonable Transformations (cont-d)

- When we start asking more people:
 - we are thus adding people who are less experienced,
 - and who may therefore be somewhat intimidated by the opinions of the top experts.
- This intimidation can be expressed in different ways.
- Some new people may be too shy to express their own opinion, so they will keep quiet; as a result:
 - if we add A people to the original N , we sill still have the same number M of people voting “yes”,
 - and the new ratio is $Y = \frac{M}{N + A}$.
- Here, $Y = a \cdot y$, where $a \stackrel{\text{def}}{=} \frac{N}{N + A}$.
- Some new people will be too shy to think on their own and will vote with the majority.

15. Reasonable Transformations (cont-d)

- So when $M > N/2$, we will have $Y = \frac{M + A}{N + A}$.
- Since $M = y \cdot N$, we will have $Y = \frac{y \cdot N + A}{N + A} = a \cdot y + b$,
where a is the same as before and $b = \frac{A}{N + A}$.
- We may also have a situation in which:
 - a certain proportion c of the new people keep quiet while
 - the others vote with the majority.
- In this case, we have $Y = \frac{M + (1 - c) \cdot A}{N + A} = a \cdot y + b$,
where $a = (1 - c) \cdot \frac{A}{N + A}$.

16. Reasonable Transformations (cont-d)

- In all these cases, we have a linear transformation

$$Y = a \cdot y + b.$$

- So, it seems reasonable to identify reasonable transformations with linear ones.
- We will call the corresponding scale-invariance L-scale-invariance (L for Linear).

17. What Membership Functions We Consider

- We consider symmetric properties, for which

$$\mu(-x) = \mu(x).$$

- So it is sufficient to consider only positive values x .
- We consider properties like “small” for which $\mu(x)$ decreases with x and $\lim_{x \rightarrow \infty} \mu(x) = 0$.
- We will call such membership functions s-membership functions (s for small).
- We say that an s-membership function $\mu(x)$ is *L-scale-invariant* if:

- for every $\lambda > 0$, there exist values $a(\lambda)$ and $b(\lambda)$
- for which $y = \mu(x)$ implies $Y = \mu(X)$, where

$$X = \lambda \cdot x \text{ and } Y = a(\lambda) \cdot y + b(\lambda).$$

18. What Membership Functions (cont-d)

- Unfortunately, this does not solve our problem:
- **Proposition 1.** *The only L -scale-invariant s -membership functions are constant functions $\mu(x) = \text{const.}$*
- What does this result mean?
- We considered two possible types of reasonable transformations of the degrees of confidence.
- They both turned out to be linear.
- This was not enough.
- So probably there are other reasonable transformations of degrees of confidence.
- How can we describe such transformations?

19. What Membership Functions (cont-d)

- Clearly, if we have a reasonable transformation, then its inverse is also reasonable.
- Also, a composition of two reasonable transformations should be a reasonable transformation too.
- So, in mathematical terms, reasonable transformations should form a *group*.
- This group should be finite-dimensional, i.e.:
 - different transformations should be uniquely determined
 - by a finite number of parameters – since in the computer, we can store only finitely many parameters.

20. What Membership Functions (cont-d)

- We also know that linear transformations are reasonable; so, we are looking for:
 - a finite-dimensional group of transformations from real numbers to real numbers
 - that contains all linear transformations.
- It is known that all such transformations are piece-wise linear: $\mu \rightarrow \frac{a \cdot \mu + b}{1 + c \cdot \mu}$.
- Thus, we arrive at the following definitions.

21. Definitions and the Main Result

- We say that an s-membership function $\mu(x)$ is *scale-invariant* if:
 - for every $\lambda > 0$, there exist $a(\lambda)$, $b(\lambda)$, and $c(\lambda)$
 - for which $y = \mu(x)$ implies $Y = \mu(X)$, where

$$X = \lambda \cdot x \text{ and } Y = \frac{a(\lambda) \cdot y + b(\lambda)}{1 + c(\lambda) \cdot y}.$$

- **Proposition 2.** *The only scale-invariant s-membership functions are distending membership functions.*
- This result explains the empirical success of distending functions.

22. Which Hedge Operations and Negation Operations Should We Select

- We would like hedging and negation operations $y = h(x)$ to be also invariant, i.e., that:
 - for each natural transformation $X = T(x)$, there should be a transformation $Y = S(y)$
 - for which $y = h(x)$ implies $Y = h(X)$.
- Now we know what are natural transformations of membership degrees – they are fractional-linear functions.
- Let us call this h-scale-invariance.
- **Proposition 3.** *The only h-scale-invariant functions are fractionally linear ones.*
- This result explains the empirical success of fractional-linear hedge operations and negation operations.

23. Proof of Proposition 1

- We will prove this result by contradiction.
- Let us assume that the function $\mu(x)$ is not a constant, and let us derive a contradiction.
- Let us substitute the expressions for X , Y , and $y = \mu(x)$ into the formula $Y = \mu(X)$.
- Then, we conclude that for every x and for every λ , we have $\mu(\lambda \cdot x) = a(\lambda) \cdot \mu(x) + b(\lambda)$.
- It is known that monotonic functions are almost everywhere differentiable; due to the above formula:
 - if a function $\mu(x)$ is differentiable at $x = x_0$,
 - it is also differentiable at any point of the type $\lambda \cdot x_0$ for every $\lambda > 0$,
 - and thus, that it is differentiable for all $x > 0$.

24. Proof of Proposition 1 (cont-d)

- Since the function $\mu(x)$ is not constant, there exist values $x_1 \neq x_2$ for which $\mu(x_1) \neq \mu(x_2)$.

- For these values, the above formula has the form

$$\mu(\lambda \cdot x_1) = a(\lambda) \cdot \mu(x_1) + b(\lambda); \quad \mu(\lambda \cdot x_2) = a(\lambda) \cdot \mu(x_2) + b(\lambda).$$

- Subtracting the two equations, we get

$$\mu(\lambda \cdot x_1) - \mu(\lambda \cdot x_2) = a(\lambda) \cdot (\mu(x_1) - \mu(x_2)), \text{ thus}$$

$$a(\lambda) = \frac{\mu(\lambda \cdot x_1) - \mu(\lambda \cdot x_2)}{\mu(x_1) - \mu(x_2)}.$$

- Since the function $\mu(x)$ is differentiable, we can conclude that the function $a(\lambda)$ is also differentiable.
- Thus, the function $b(\lambda) = \mu(\lambda \cdot x) - a(\lambda) \cdot \mu(x)$ is differentiable too.
- So, all three functions $\mu(x)$, $a(\lambda)$, and $b(\lambda)$ are differentiable.

25. Proof of Proposition 1 (cont-d)

- So, we can differentiate both sides of the equality

$$\mu(\lambda \cdot x) = a(\lambda) \cdot \mu(x) + b(\lambda) \text{ with respect to } \lambda.$$

- If we substitute $\lambda = 1$, we get $x \cdot \mu'(x) = A \cdot \mu(x) + B$, where we denoted $A \stackrel{\text{def}}{=} a'(1)$, $B \stackrel{\text{def}}{=} b'(1)$.

- Here, $\mu'(x)$, as usual, indicates the derivative.

- Thus, $x \cdot \frac{d\mu}{dx} = A \cdot \mu + B$.

- We cannot have $A = 0$ and $B = 0$, since then $\mu'(x) = 0$ and $\mu(x)$ would be a constant.

- Thus, in general, the expression $A \cdot \mu + B$ is not 0, so

$$\frac{d\mu}{A \cdot \mu + B} = \frac{dx}{x}.$$

- If $A = 0$, then integration leads to $\frac{1}{B} \cdot \mu(x) = \ln(x) + c$, where c_0 is the integration constant.

26. Proof of Proposition 1 (cont-d)

- Thus, $\mu(x) = B \cdot \ln(x) + B \cdot c_0$.
- This expression has negative values for some x , while all the values $\mu(x)$ are in the interval $[0, 1]$.
- So, this case is impossible.
- If $A \neq 0$, then we have $d(A \cdot \mu + B) = A \cdot d\mu$, hence

$$\frac{d(A \cdot \mu + B)}{A \cdot \mu + B} = A \cdot \frac{dx}{x}.$$

- Integration leads to $\ln(A \cdot \mu(x) + B) = A \cdot \ln(x) + c_0$.
- By applying $\exp(z)$ to both sides, we get $A \cdot \mu(x) + B = \exp(c_0) \cdot x^A$, i.e., $\mu(x) = A^{-1} \cdot \exp(c_0) \cdot x^A - B/A$.
- This expression tends to infinity either for $x \rightarrow \infty$ (if $A > 0$) or for $x \rightarrow 0$ (if $A < 0$).
- In both cases, we get a contradiction with our assumption that $\mu(x)$ is within the interval $[0, 1]$. Q.E.D.

27. Proof of Proposition 2

- Let us substitute the expressions for X , Y , and $y = \mu(x)$ into the formula $Y = \mu(X)$.
- Then, we conclude that for every x and for every λ :

$$\mu(\lambda \cdot x) = \frac{a(\lambda) \cdot \mu(x) + b(\lambda)}{1 + c(\lambda) \cdot \mu(x)}.$$

- Similarly to the previous proof, we can conclude that the function $\mu(x)$ is differentiable for all $x > 0$.
- Multiplying both sides of the above equality by the denominator, we conclude that

$$\mu(\lambda \cdot x) + c(\lambda) \cdot \mu(x) \cdot \mu(\lambda \cdot x) = a(\lambda) \cdot \mu(x) + b(\lambda).$$

- So, for three different values x_i , we have the following three equations:

$$\mu(\lambda \cdot x_i) + c(\lambda) \cdot \mu(x_i) \cdot \mu(\lambda \cdot x_i) = a(\lambda) \cdot \mu(x_i) + b(\lambda), \quad i = 1, 2, 3.$$

28. Proof of Proposition 2 (cont-d)

- We thus have a system of three linear equations for three unknowns $a(\lambda)$, $b(\lambda)$, and $c(\lambda)$.
- By Cramer's rule:
 - the solution to such a system
 - is a rational (hence differentiable) function of the coefficients and the right-hand sides.
- So, since $\mu(x)$ is differentiable, we can conclude that $a(\lambda)$, $b(\lambda)$, and $c(\lambda)$ are differentiable.
- All the functions $\mu(x)$, $a(\lambda)$, $b(\lambda)$, and $c(\lambda)$ are differentiable.
- So, we can differentiate both sides of the above formula with respect to λ .
- Let us substitute $\lambda = 1$ and take into account that for $\lambda = 1$, we have $a(1) = 1$ and $b(1) = c(1) = 0$.

29. Proof of Proposition 2 (cont-d)

- Then, we get $x \cdot \frac{d\mu}{dx} = A \cdot \mu + B - C \cdot \mu^2$, where A and B are the same as in the previous proof and $C \stackrel{\text{def}}{=} c'(1)$.
- For $x \rightarrow \infty$, we have $\mu(x) \rightarrow 0$, so $\mu'(x) \rightarrow 0$, and thus $B = 0$ and $x \cdot \frac{d\mu}{dx} = A \cdot \mu - C \cdot \mu^2$.
- So, $\frac{d\mu}{B \cdot \mu - C \cdot \mu^2} = \frac{dx}{x}$.
- As we have shown in the previous proof, we cannot have $C = 0$, so $C \neq 0$.
- One can easily see that

$$\frac{1}{\mu - \frac{B}{C}} - \frac{1}{\mu} = \frac{\frac{B}{C}}{\mu \cdot \left(\mu - \frac{B}{C} \right)} = \frac{-B}{B \cdot \mu - C \cdot \mu^2}.$$

30. Proof of Proposition 2 (cont-d)

- Thus, by multiplying the equality $\frac{d\mu}{B \cdot \mu - C \cdot \mu^2} = \frac{dx}{x}$ by $-B$, we get: $\frac{d\mu}{\mu - \frac{C}{B}} - \frac{d\mu}{\mu} = -B \cdot \frac{dx}{x}$.

- Integrating both sides, we get

$$\ln\left(\mu(x) - \frac{B}{C}\right) - \ln(\mu) = -B \cdot \ln(x) + c_0.$$

- By applying $\exp(z)$ to both sides, we get

$$\frac{\mu(x) - \frac{B}{C}}{\mu(x)} = C_0 \cdot x^{-B}. \text{ so } 1 - \frac{B/C}{\mu} = C_0 \cdot x^{-B}.$$

- Hence $\frac{B/C}{\mu} = 1 - C_0 \cdot x^{-B}$ and $\mu(x) = \frac{B/C}{1 - C_0 \cdot x^{-B}}$.

- From the condition that $\mu(0) = 1$, we conclude that $B < 0$ and $B/C = 1$.

31. Proof of Proposition 2 (cont-d)

- From $\mu(x) \leq 1$, we conclude that $C_0 < 0$.
- So, we get the desired formula $\mu(x) = \frac{1}{1 + |C_0| \cdot x^{|B|}}$.
- The proposition is proven.

32. Proof of Proposition 3

- For constant functions the statement is trivial.
- Therefore, it is sufficient to prove for non-constant functions $h(x)$.
- Similarly to the previous proof, we can prove that the function $h(x)$ is differentiable.
- Let $x \in D$, and let λ and x_0 from an open neighborhood of 1 and 0 respectively be such that

$$\lambda \cdot x \in D \text{ and } x + x_0 \in D.$$

- Since the function $h(x)$ is h-scale-invariant, there exist fractional-linear transformations for which

$$h(x + x_0) = \frac{a(x_0) \cdot h(x) + b(x_0)}{1 + c(x_0) \cdot h(x)} \text{ and}$$

$$h(\lambda \cdot x) = \frac{d(\lambda) \cdot h(x) + e(\lambda)}{1 + f(\lambda) \cdot h(x)}.$$

33. Proof of Proposition 3 (cont-d)

- Similarly to the previous proof, we can prove that the functions $a(x_0), \dots$, are differentiable.
- So, we can differentiate the λ -formula with respect to λ and take $\lambda = 1$, then we get:

$$x \cdot h' = D \cdot h + E - F \cdot h^2.$$

- Similarly, differentiating the h_0 -formula with respect to x_0 and taking $x_0 = 0$, we get:

$$h' = A \cdot h + B - C \cdot h^2.$$

- Let us consider two cases: $C \neq 0$ and $C = 0$.
- Let us first consider the case when $C \neq 0$.
- By completing the square, we get $h' = A \cdot h + B - C \cdot h^2 = \widehat{A} - C \cdot (h - h_0)^2$ for some \widehat{A} and h_0 , i.e.,

$$h' = \widehat{A} - C \cdot H^2, \text{ where } H \stackrel{\text{def}}{=} h - h_0.$$

34. Proof of Proposition 3 (cont-d)

- Substituting $h = H + h_0$ into the right-hand side, we conclude that

$$x \cdot h' = \widehat{D} \cdot H + \widehat{E} - F \cdot H^2 \text{ for some } \widehat{D} \text{ and } \widehat{E}.$$

- Dividing the two equations, we get

$$x = \frac{\widehat{D} \cdot H + \widehat{E} - F \cdot H^2}{\widehat{A} - C \cdot H^2}, \text{ so } \frac{dx}{dH} =$$

$$\frac{(\widehat{D} - 2F \cdot H)(\widehat{A} - C \cdot H^2) - (\widehat{D} \cdot H + \widehat{E} - F \cdot H^2)(-2C \cdot H)}{(\widehat{A} - C \cdot H^2)^2} =$$
$$\frac{\widehat{A} \cdot \widehat{D} - 2(\widehat{A} \cdot F - C \cdot \widehat{E}) \cdot H + C \cdot \widehat{D} \cdot H^2}{(\widehat{A} - C \cdot H^2)^2}.$$

- On the other hand,

$$\frac{dx}{dH} = \frac{1}{\frac{dH}{dx}} = \frac{1}{\widehat{A} - C \cdot H^2}.$$

35. Proof of Proposition 3 (cont-d)

- The right-hand sides of these two formulas must be equal, so for all H , we have

$$\widehat{A} \cdot \widehat{D} - 2(\widehat{A} \cdot F - C \cdot \widehat{E}) \cdot H + C \cdot \widehat{D} \cdot H^2 = \widehat{A} - C \cdot H^2.$$

- Since the two polynomials of H are equal, the coefficients at 1, H , and H^2 must coincide.
- Comparing the coefficients at H^2 , we get $C \cdot \widehat{D} = -C$.
- Since $C \neq 0$, we conclude that $\widehat{D} = -1$.
- Comparing the coefficients at 1, we get $\widehat{A} \cdot \widehat{D} = \widehat{A}$, i.e., $-\widehat{A} = \widehat{A}$ and thus $\widehat{A} = 0$.
- Comparing the coefficients at H and taking into account that $\widehat{A} = 0$, we get $0 = \widehat{A} \cdot F - C \cdot \widehat{E} = -C \cdot \widehat{E}$.
- Since $C \neq 0$, this implies $\widehat{E} = 0$.

36. Proof of Proposition 3 (cont-d)

- So, the above formula for x takes the form

$$x = \frac{\widehat{D} \cdot H - F \cdot H^2}{-C \cdot H^2} = \frac{\widehat{D} - F \cdot H}{-C \cdot H}.$$

- Thus x is a fractional linear function of H .
- Hence H (and therefore $h = H + h_0$) is also a fractional linear function of x .
- Let us now consider the case when $C = 0$.
- Then, $h' = A \cdot h + B$ and $x \cdot h' = D \cdot h + E - F \cdot h^2$, so:

$$x = \frac{x \cdot h'}{h'} = \frac{D \cdot h + E - F \cdot h^2}{A \cdot h + B}.$$

- If $F = 0$, then x is a fractional linear function of $h(x)$ and hence, h is also a fractional-linear function of x .
- So, it is sufficient to consider the case when $F \neq 0$.

37. Proof of Proposition 3 (cont-d)

- In this case, by completing the square, we can find constants \widehat{D} , h_0 , and \widehat{B} for which, for $H = h - h_0$:

$$x \cdot h' = D \cdot h + E - F \cdot h^2 = \widetilde{D} - F \cdot H^2 \text{ and}$$
$$h' = A \cdot h + B = A \cdot H + \widehat{B}.$$

- Dividing the first equation by the second one, we have

$$x = \frac{\widetilde{D} - F \cdot H^2}{A \cdot H + \widehat{B}}, \text{ thus}$$

$$\frac{dx}{dH} = \frac{(-2F \cdot H) \cdot (A \cdot H + \widehat{B}) - (\widetilde{D} - F \cdot H^2) \cdot A}{(A \cdot H + \widehat{B})^2}$$
$$= \frac{-A \cdot \widehat{D} - 2\widehat{B} \cdot F \cdot H - A \cdot F \cdot H^2}{(A \cdot H + \widehat{B})^2}.$$

- On the other hand, $\frac{dx}{dH} = \frac{1}{\frac{dH}{dx}} = \frac{1}{A \cdot H + \widehat{B}}.$

38. Proof of Proposition 3 (cont-d)

- By equating the two expressions for the derivative and multiplying both sides by $(A \cdot H + \widehat{B})^2$, we get:

$$-A \cdot \widehat{D} - 2\widehat{B} \cdot F \cdot H - A \cdot F \cdot H^2 = A \cdot H + \widehat{B}.$$

- Thus $A \cdot F = 0$, $A = -2\widehat{B} \cdot F$, and $-A \cdot \widehat{D} = \widehat{B}$.
- If $A = 0$, then we have $\widehat{B} = 0$, so $h' = 0$ and h is a constant.
- However, we consider the case when the function $h(x)$ is not a constant.
- Thus, $A \neq 0$, hence $F = 0$, and the above formula describes x as a fractional-linear function of H .
- Both for $C \neq 0$ and $C = 0$, x is fractionally linear in H (hence in h).
- Since the inverse of a fractional linear is fractional linear, the function $h(x)$ is also fractional linear. Q.E.D.

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