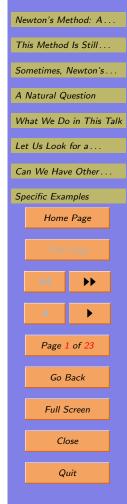
# Equations for Which Newton's Method Never Works: Pedagogical Examples

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#### 1. Newton's Method: A Brief Reminder

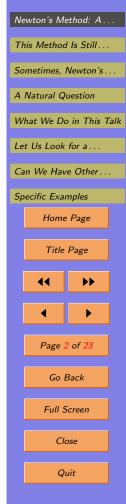
- This method is based on the fact that the derivative f'(x) is defined as  $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ .
- This means that for small h, the derivative is approximately equal to this ratio.
- In this approximation,  $f'(x) \approx \frac{f(x+h) f(x)}{h}$ .
- Multiplying both sides by h, we get

$$f'(x) \cdot h \approx f(x+h) - f(x).$$

• Thus, adding f(x) to both sides, we get

$$f(x+h) \approx f(x) + h \cdot f'(x)$$
.

• Suppose that we know some approximation  $x_k$  to the desired value x.



# 2. Newton's Method (cont-d)

- For this approximation,  $f(x_k)$  is not exactly 0, so:
  - to make the value f(x) closer to 0,
  - it is therefore reasonable to make a small modification of the current approximation, i.e., take

$$x_{k+1} = x_k + h.$$

- For this new value, according to the above formula, we have  $f(x_{k+1}) \approx f(x_k) + h \cdot f'(x_k)$ .
- We want to get the value  $f(x_{k+1})$  as close to 0 as possible.
- So, it is therefore reasonable to take h for which

$$f(x_k) + h \cdot f'(x_k) = 0$$
, i.e.,  $h = -\frac{f(x_k)}{f'(x_k)}$ .



# 3. Newton's Method (cont-d)

• Thus, the next approximation  $x_{k+1} = x_k + h$  is:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

- This is exactly what Newton has proposed.
- If this method converges precisely in the sense that we have  $x_{k+1} = x_k$ , then  $f(x_k) = 0$ .
- So,  $x_k$  is the desired solution.
- If this method converges approximately, i.e.,
  - if the difference  $x_{k+1} x_k$  is very small,
  - then we conclude that the value  $f(x_k)$  is also very small, and
  - thus, we have a good approximation to the desired solution.



# 4. This Method Is Still Actively Used to Solve Equations

- This method being centuries old.
- However, it is still used to solve many practical problems.
- For example:
  - this is how most computers compute the square root of a given number a, i.e.,
  - how computers compute the solution to the equation f(x) = 0 with  $f(x) = x^2 - a$ .
- For this function f(x), we have f'(x) = 2x, thus Newton's formula takes the form  $x_{k+1} = x_k \frac{x_k^2 a}{2x_k}$ .
- This formula can be simplified if we take into account that  $\frac{x_k^2}{2x_k} = \frac{x_k}{2}$ , thus  $x_{k+1} = \frac{1}{2} \cdot \left(x_k + \frac{a}{x_k}\right)$ .

Newton's Method: A... This Method Is Still Sometimes, Newton's.. A Natural Question What We Do in This Talk Let Us Look for a . . . Can We Have Other . . . Specific Examples Home Page Title Page **>>** Page 5 of 23 Go Back Full Screen Close

Quit

- The simplified formula is indeed faster to compute than the original formula:
  - both formulas require one division,
  - but the original also requires one multiplication (to compute  $x_k^2$ ) and two subtractions,
  - while the new formula needs only one addition.
- Both formula need multiplication or division by 2.
- However for binary numbers, this is trivial just shifting by 1 bit to the left or to the right.
- The resulting iterative process converges fast.
- For example, to compute  $\sqrt{2}$ , we can start with  $x_0 = 1$  and get  $x_1 = \frac{1}{2} \cdot \left(1 + \frac{2}{1}\right) = 1.5$ .



- Then  $x_2 = \frac{1}{2} \cdot \left( 1.5 + \frac{2}{1.5} \right) = 1.4166 \dots$
- In only two iterations, we already have the first three digits of the correct answer  $\sqrt{2} = 1.414...$
- Newton's method also lies behind the way computers divide.
- To be more precise, computes compute the ratio  $\frac{a}{b}$  by:
  - first computing the inverse  $\frac{1}{h}$ , and
  - then multiplying a by this inverse.



- To compute the inverse, computers:
  - contain a table of pre-computed values of the inverse for several fixed values  $B_i$ , and,
  - then, for  $b \approx B_i$ , use the recorded inverse  $\frac{1}{B_i}$  as the first approximation  $x_0$  in the Newton's method.
- In this case, the desired equation has the form  $b \cdot x 1 = 0$ , i.e., here  $f(x) = b \cdot x 1$ .
- The actual derivative f'(x) is equal to b.
- So, ideally we should have  $x_{k+1} = x_k \frac{1}{b} \cdot (b \cdot x_j 1)$ .



- This may sound reasonable, but:
  - the whole purpose of this algorithm is to compute the inverse value  $\frac{1}{b}$ ,
  - so we do not know it yet and thus, we cannot use the above formula directly.
- What we do know, at this stage, is the current approximation  $x_k$  to the desired inverse value  $\frac{1}{h}$ .
- So, a natural idea is to use  $x_k$  instead of the inverse value in the Netwon's formula.
- Then, we get exactly the form of Newton's method that computers use to compute the inverse:

$$x_{k+1} = x_k - (b \cdot x_k - 1) \cdot x_k.$$



- Reminder:  $x_{k+1} = x_k (b \cdot x_k 1) \cdot x_k$ .
- It should be mentioned that, similar to  $\sqrt{a}$ , this expression can also be further simplified, e.g., to

$$x_{k+1} = x_k \cdot (2 - b \cdot x_k).$$

Both formulas require two multiplications.

- However, the simplified formula is slightly faster to compute since:
  - this formula requires only one subtraction, while
  - the original formula requires two subtractions.



# 10. Sometimes, Newton's Method Does Not Work

- While Newton's method is efficient, there are examples when it does not work.
- Such examples are usually given in textbooks, explaining the need for alternative techniques.
- Sometimes, this happens because the values  $x_k$  diverge, i.e.:
  - become larger and larger with each iteration,
  - never converging to anything.
- Sometime, this happens because the values  $x_k$  from a loop:
  - we get  $x_0, ..., x_{k-1}$ , and
  - then we again get  $x_k = x_0$ ,  $x_{k+1} = x_1$ , etc.,
  - and the process also never converges.



#### 11. A Natural Question

- The textbook examples usually show that:
  - whether Newton's method is successful
  - depends on how close is the initial approximation  $x_0$  to the actual solution x.
- Specifically:
  - if  $x_0$  is close to x, then usually, Newton's method converges, while
  - if the initial approximation  $x_0$  is far away from x, Newton's method starts diverging.
- A natural question that students sometimes ask is:
  - whether this is always the case, or
  - whether there are examples when Newton's method never converges.



#### 12. What We Do in This Talk

- In this talk, we provide examples when Newton's method:
  - practically never converges,
  - no matter what initial approximation  $x_0$  we take.
- It only converges when we take the solution x as the first approximation:  $x_0 = x$ .



# 13. Let Us Look for a Simple Example

- Let us first look for examples in which the equation f(x) = 0 has only one solution.
- For simplicity, let us assume that the desired solution is x = 0.
- Again, for simplicity, let us consider odd functions f(x), i.e. functions for which f(-x) = -f(x).
- Let us also consider the simplest possible case when the Newton's method does not converge:
  - when the iterations  $x_k$  form a loop, and
  - let us consider the simplest possible loop:  $x_0, x_1 \neq x_0$ , and  $x_2 = x_0$ .



# 14. How to Come up With Such a Simple Example

- In general, the closer  $x_0$  to the solution, the closer  $x_1$  will be.
- If  $x_1$  was on the same side of the solution as  $x_0$ , then:
  - if  $x_1 < x_0$ , we would eventually have convergence, and
  - if  $x_1 > x_0$ , we would have divergence.
- However, we want a loop.
- Thus,  $x_1$  should be on the other side of  $x_0$ .
- Since the function f(x) is odd, the dependence of  $x_2$  on  $x_1$  is the same as the dependence of  $x_1$  on  $x_0$ , so:
  - if  $|x_1| < |x_0|$ , we would have convergence, and
  - if  $|x_1| > |x_0|$ , we would have divergence.
- The only way to get a loop is thus to have  $|x_1| = |x_0|$ .



# 15. Towards a Simple Example (cont-d)

- Since the values  $x_0$  and  $x_1$  are on the other solution of the solution x = 0, this means that  $x_1 = -x_0$ .
- The Newton's formula is  $x_1 = x_0 \frac{f(x_0)}{f'(x_0)}$ .
- Thus, the desired equality  $x_1 = -x_0$  means that

$$-x_0 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

- We want to have an example in which the Newton's process will loop for all  $x_0 \neq 0$ .
- Thus, for all  $x \neq 0$ , we should have  $-x = x \frac{f(x)}{f'(x)}$ .

Newton's Method: A... This Method Is Still . . . Sometimes, Newton's... A Natural Question What We Do in This Talk Let Us Look for a . . . Can We Have Other . . . Specific Examples Home Page Title Page **>>** 44 Page 16 of 23 Go Back Full Screen Close

Quit

# 16. Let Us Solve This Equation

• By moving the ratio the left-hand side and -x to the right-hand side, we get

$$2x = \frac{f}{df}$$
, i.e.  $2x = \frac{f \cdot dx}{df}$ .

- We can separate x and f if we multiply both sides by df and divide both sides by f and by 2x.
- As a result, we get  $\frac{df}{f} = \frac{dx}{2x}$ .
- Integrating both sides, we get  $\ln(f) = \frac{1}{2} \cdot \ln(x) + C$ , where C is the integration constant.
- Applying  $\exp(z)$  to both sides of this equality, we get  $f(x) = c \cdot \sqrt{x}$ , where  $c \stackrel{\text{def}}{=} \exp(C)$ .



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# 17. Let Us Solve This Equation (cont-d)

• Since we want an odd function, we thus get

$$f(x) = c \cdot sign(x) \cdot \sqrt{|x|}$$
, where:

- $-\operatorname{sign}(x) = 1 \text{ for } x > 0 \text{ and }$
- $-\operatorname{sign}(x) = -1 \text{ for } x < 0.$
- Of course, if we shift the function by some value a, we get a similar behavior.
- Thus, in general, we have a 2-parametric family of functions for which the Newton's method always loops:

$$f(x) = c \cdot \operatorname{sign}(x) \cdot \sqrt{|x - a|}.$$

- Interestingly:
  - the simplest example  $f(x) = \sqrt{x}$  on which Newton's method never works
  - is exactly inverse to the simplest example  $f(x) = x^2$  when it works perfectly.



# 18. Can We Have Other Examples?

- Suppose that we have a non-negative function f(x) defined for non-negative x:
  - for which f(0) = 0 and
  - for which, for each  $x_0 > 0$ , the next step of the Newton's method leads to the value  $x_1 < 0$ :

$$x - \frac{f(x)}{f'(x)} < 0.$$

- This inequality can be reformulated as f'/f < x, i.e., as  $\frac{\ln(f)}{\ln(x)} < 1$ .
- So, in the log-log scale, the slope is always smaller than 1.
- We will also assume that the difference  $x \frac{f(x)}{f'(x)}$  monotonically depends on x.



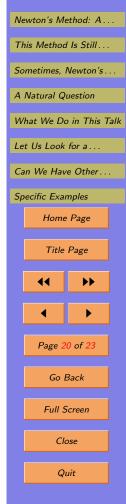
# 19. How to Design Such Looping Examples

- We would like to extend f(x) to negative x in such a way that the Newton's process will always loop.
- For convenience, let us denote, for each x > 0,  $F(x) \stackrel{\text{def}}{=} -f(-x)$ , where f(-x) is the desired extension.
- Then, for x < 0, we have f(x) = -F(-x).
- When we start with the initial value x > 0, the next iteration is -y, where we denoted

$$y = \frac{f(x)}{f'(x)} - x.$$

• Then, if we want the simplest loop, on the next iteration, we should get back the value x:

$$x = (-y) - \frac{f'(-y)}{f'(-y)}.$$



# 20. Designing a Looping Examples (cont-d)

• Substituting f(x) = -F(-x) into this equality, we get  $x = \frac{F(y)}{F'(y)} - y$ , i.e., equivalently,

$$\frac{F'(y)}{F(y)} = \frac{1}{x+y}$$
 thus,  $F'(y) = \frac{F(y)}{x+y}$ .

- We thus have a differential equation that enables us:
  - to reconstruct, step-by-step, the desired function F(y) and thus,
  - to reconstruct the desired extension of f(x) to negative values.



# 21. Specific Examples

- When  $f(x) = x^a$  for some a > 0, the log-log inequality implies that a < 1.
- One can check that in this case, we can take

$$F(y) = y^{1-a}.$$

 $\bullet$  So, we extend this function to negative values x as

$$f(x) = -|x|^{1-a}$$
.

• In particular, for a = 1/2, we get the above square root example.



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Quit