Equations for Which Newton's Method Never Works: Pedagogical Examples

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1. Newton's Method: A Brief Reminder

- This method is based on the fact that the derivative $f'(x)$ is defined as $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.
- This means that for small h, the derivative is approximately equal to this ratio.
- In this approximation, $f'(x) \approx \dfrac{f(x+h)-f(x)}{h}$.
- Multiplying both sides by h, we get
$$f'(x)\cdot h \approx f(x+h)-f(x).$$
- Thus, adding $f(x)$ to both sides, we get
$$f(x+h) \approx f(x) + h\cdot f'(x).$$
- Suppose that we know some approximation x_k to the desired value x.

2. Newton's Method (cont-d)

- For this approximation, $f(x_k)$ is not exactly 0, so:
  - to make the value $f(x)$ closer to 0,
  - it is therefore reasonable to make a small modification of the current approximation, i.e., take

$$x_{k+1} = x_k + h.$$

- For this new value, according to the above formula, we have $f(x_{k+1}) \approx f(x_k) + h \cdot f'(x_k)$.
- We want to get the value $f(x_{k+1})$ as close to 0 as possible.
- So, it is therefore reasonable to take h for which

$$f(x_k) + h \cdot f'(x_k) = 0, \text{ i.e., } h = -\frac{f(x_k)}{f'(x_k)}.$$

3. Newton's Method (cont-d)

- Thus, the next approximation $x_{k+1} = x_k + h$ is:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

- This is exactly what Newton has proposed.
- If this method converges precisely – in the sense that we have $x_{k+1} = x_k$, then $f(x_k) = 0$.
- So, x_k is the desired solution.
- If this method converges approximately, i.e.,
  - if the difference $x_{k+1} - x_k$ is very small,
  - then we conclude that the value $f(x_k)$ is also very small, and
  - thus, we have a good approximation to the desired solution.

4. This Method Is Still Actively Used to Solve Equations

- This method being centuries old.
- However, it is still used to solve many practical problems.
- For example:
  - this is how most computers compute the square root of a given number a, i.e.,
  - how computers compute the solution to the equation $f(x) = 0$ with $f(x) = x^2 - a$.
- For this function $f(x)$, we have $f'(x) = 2x$, thus Newton's formula takes the form $x_{k+1} = x_k - \dfrac{x_k^2 - a}{2x_k}$.
- This formula can be simplified if we take into account that $\dfrac{x_k^2}{2x_k} = \dfrac{x_k}{2}$, thus $x_{k+1} = \dfrac{1}{2} \cdot \left(x_k + \dfrac{a}{x_k}\right)$.

5. This Method Is Actively Used (cont-d)

- The simplified formula is indeed faster to compute than the original formula:
  - both formulas require one division,
  - but the original also requires one multiplication (to compute x_k^2) and two subtractions,
  - while the new formula needs only one addition.
- Both formula need multiplication or division by 2.
- However for binary numbers, this is trivial – just shifting by 1 bit to the left or to the right.
- The resulting iterative process converges fast.
- For example, to compute $\sqrt{2}$, we can start with $x_0 = 1$ and get $x_1 = \frac{1}{2} \cdot \left(1 + \frac{2}{1}\right) = 1.5$.

6. This Method Is Actively Used (cont-d)

- Then $x_2 = \frac{1}{2} \cdot \left(1.5 + \frac{2}{1.5}\right) = 1.4166\ldots$
- In only two iterations, we already have the first three digits of the correct answer $\sqrt{2} = 1.414\ldots$
- Newton's method also lies behind the way computers divide.
- To be more precise, computes compute the ratio $\frac{a}{b}$ by:
  - first computing the inverse $\frac{1}{b}$, and
  - then multiplying a by this inverse.

7. This Method Is Actively Used (cont-d)

- To compute the inverse, computers:
  - contain a table of pre-computed values of the inverse for several fixed values B_i, and,
  - then, for $b \approx B_i$, use the recorded inverse $\dfrac{1}{B_i}$ as the first approximation x_0 in the Newton's method.
- In this case, the desired equation has the form $b \cdot x - 1 = 0$, i.e., here $f(x) = b \cdot x - 1$.
- The actual derivative $f'(x)$ is equal to b.
- So, ideally we should have $x_{k+1} = x_k - \dfrac{1}{b} \cdot (b \cdot x_j - 1)$.

8. This Method Is Actively Used (cont-d)

- This may sound reasonable, but:
  - the whole purpose of this algorithm is to compute the inverse value $\frac{1}{b}$,
  - so we do not know it yet and thus, we cannot use the above formula directly.
- What we *do* know, at this stage, is the current approximation x_k to the desired inverse value $\frac{1}{b}$.
- So, a natural idea is to use x_k instead of the inverse value in the Netwon's formula.
- Then, we get exactly the form of Newton's method that computers use to compute the inverse:

$$x_{k+1} = x_k - (b \cdot x_k - 1) \cdot x_k.$$

9. This Method Is Actively Used (cont-d)

- Reminder: $x_{k+1} = x_k - (b \cdot x_k - 1) \cdot x_k$.
- It should be mentioned that, similar to $\sqrt{a}$, this expression can also be further simplified, e.g., to

$$x_{k+1} = x_k \cdot (2 - b \cdot x_k).$$

  Both formulas require two multiplications.
- However, the simplified formula is slightly faster to compute since:
  - this formula requires only one subtraction, while
  - the original formula requires two subtractions.

10. Sometimes, Newton's Method Does Not Work

- While Newton's method is efficient, there are examples when it does not work.
- Such examples are usually given in textbooks, explaining the need for alternative techniques.
- Sometimes, this happens because the values x_k diverge, i.e.:
  - become larger and larger with each iteration,
  - never converging to anything.
- Sometime, this happens because the values x_k from a loop:
  - we get $x_0, \ldots, x_{k-1}$, and
  - then we again get $x_k = x_0$, $x_{k+1} = x_1$, etc.,
  - and the process also never converges.

11. A Natural Question

- The textbook examples usually show that:
  - whether Newton's method is successful
  - depends on how close is the initial approximation x_0 to the actual solution x.
- Specifically:
  - if x_0 is close to x, then usually, Newton's method converges, while
  - if the initial approximation x_0 is far away from x, Newton's method starts diverging.
- A natural question – that students sometimes ask – is:
  - whether this is always the case, or
  - whether there are examples when Newton's method never converges.

12. What We Do in This Talk

- In this talk, we provide examples when Newton's method:
  - practically never converges,
  - no matter what initial approximation x_0 we take.
- It only converges when we take the solution x as the first approximation: $x_0 = x$.

13. Let Us Look for a Simple Example

- Let us first look for examples in which the equation $f(x) = 0$ has only one solution.
- For simplicity, let us assume that the desired solution is $x = 0$.
- Again, for simplicity, let us consider odd functions $f(x)$, i.e. functions for which $f(-x) = -f(x)$.
- Let us also consider the simplest possible case when the Newton's method does not converge:
  - when the iterations x_k form a loop, and
  - let us consider the simplest possible loop: x_0, $x_1 \neq x_0$, and $x_2 = x_0$.

14. How to Come up With Such a Simple Example

- In general, the closer x_0 to the solution, the closer x_1 will be.
- If x_1 was on the same side of the solution as x_0, then:
  - if $x_1 < x_0$, we would eventually have convergence, and
  - if $x_1 > x_0$, we would have divergence.
- However, we want a loop.
- Thus, x_1 should be on the other side of x_0.
- Since the function $f(x)$ is odd, the dependence of x_2 on x_1 is the same as the dependence of x_1 on x_0, so:
  - if $|x_1| < |x_0|$, we would have convergence, and
  - if $|x_1| > |x_0|$, we would have divergence.
- The only way to get a loop is thus to have $|x_1| = |x_0|$.

15. Towards a Simple Example (cont-d)

- Since the values x_0 and x_1 are on the other solution of the solution $x = 0$, this means that $x_1 = -x_0$.
- The Newton's formula is $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.
- Thus, the desired equality $x_1 = -x_0$ means that

$$-x_0 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

- We want to have an example in which the Newton's process will loop for all $x_0 \neq 0$.
- Thus, for all $x \neq 0$, we should have $-x = x - \dfrac{f(x)}{f'(x)}$.

16. Let Us Solve This Equation

- By moving the ratio the left-hand side and $-x$ to the right-hand side, we get

$$2x = \frac{f}{\frac{df}{dx}}, \text{ i.e. }, 2x = \frac{f \cdot dx}{df}.$$

- We can separate x and f if we multiply both sides by df and divide both sides by f and by $2x$.
- As a result, we get $\frac{df}{f} = \frac{dx}{2x}$.
- Integrating both sides, we get $\ln(f) = \frac{1}{2} \cdot \ln(x) + C$, where C is the integration constant.
- Applying $\exp(z)$ to both sides of this equality, we get $f(x) = c \cdot \sqrt{x}$, where $c \stackrel{\text{def}}{=} \exp(C)$.

17. Let Us Solve This Equation (cont-d)

- Since we want an odd function, we thus get
$$f(x) = c \cdot \text{sign}(x) \cdot \sqrt{|x|}, \text{ where:}$$
  - $\text{sign}(x) = 1$ for $x > 0$ and
  - $\text{sign}(x) = -1$ for $x < 0$.
- Of course, if we shift the function by some value a, we get a similar behavior.
- Thus, in general, we have a 2-parametric family of functions for which the Newton's method always loops:
$$f(x) = c \cdot \text{sign}(x) \cdot \sqrt{|x - a|}.$$
- Interestingly:
  - the simplest example $f(x) = \sqrt{x}$ on which Newton's method never works
  - is exactly inverse to the simplest example $f(x) = x^2$ when it works perfectly.

18. Can We Have Other Examples?

- Suppose that we have a non-negative function $f(x)$ defined for non-negative x:
  - for which $f(0) = 0$ and
  - for which, for each $x_0 > 0$, the next step of the Newton's method leads to the value $x_1 < 0$:

$$x - \frac{f(x)}{f'(x)} < 0.$$

- This inequality can be reformulated as $f'/f < x$, i.e., as $\frac{\ln(f)}{\ln(x)} < 1$.
- So, in the log-log scale, the slope is always smaller than 1.
- We will also assume that the difference $x - \frac{f(x)}{f'(x)}$ monotonically depends on x.

19. How to Design Such Looping Examples

- We would like to extend $f(x)$ to negative x in such a way that the Newton's process will always loop.
- For convenience, let us denote, for each $x > 0$, $F(x) \stackrel{\text{def}}{=} -f(-x)$, where $f(-x)$ is the desired extension.
- Then, for $x < 0$, we have $f(x) = -F(-x)$.
- When we start with the initial value $x > 0$, the next iteration is $-y$, where we denoted

$$y = \frac{f(x)}{f'(x)} - x.$$

- Then, if we want the simplest loop, on the next iteration, we should get back the value x:

$$x = (-y) - \frac{f'(-y)}{f'(-y)}.$$

20. Designing a Looping Examples (cont-d)

- Substituting $f(x) = -F(-x)$ into this equality, we get $x = \dfrac{F(y)}{F'(y)} - y$, i.e., equivalently,

$$\frac{F'(y)}{F(y)} = \frac{1}{x+y} \text{ thus, } F'(y) = \frac{F(y)}{x+y}.$$

- We thus have a differential equation that enables us:
  - to reconstruct, step-by-step, the desired function $F(y)$ and thus,
  - to reconstruct the desired extension of $f(x)$ to negative values.

21. Specific Examples

- When $f(x) = x^a$ for some $a > 0$, the log-log inequality implies that $a < 1$.
- One can check that in this case, we can take

$$F(y) = y^{1-a}.$$

- So, we extend this function to negative values x as

$$f(x) = -|x|^{1-a}.$$

- In particular, for $a = 1/2$, we get the above square root example.

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