Why Homogeneous Membranes Lead to Optimal Water Desalination: A Possible Explanation

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1. Formulation of the Problem

- One of the most efficient desalinization techniques is the use of biological membranes.
- Traditionally, researchers believed that the efficiency of a membrane is determined:
  - by the average values of relevant quantities,
  - such as the average density of the proteins forming the membrane.
- It was known that:
  - the knowledge of all these average values
  - enables us to only approximately estimate the membrane's efficiency.
- Membranes with the same average values of the corr. quantities may have different efficiencies.

2. Formulation of the Problem (cont-d)

- A recent paper:
  - used innovative nano-imaging and nano-manipulating techniques
  - to analyze and control the nano-structure of different membranes.
- The resulting analysis shows that:
  - the difference between efficiencies of different membranes with the same average values
  - can be explained by the fact that different membranes have different degrees of homogeneity:
  - specifically, homogenized membranes can be 50% more efficient than the usual ones.

3. Formulation of the Problem (cont-d)

- According to this paper,
  - the new phenomenon can be explained
  - by the fact that density fluctuations are detrimental to water transport.
- This paper also mentions that there may be other factors affecting this phenomenon.
- In this talk, we provide a general symmetry-based explanation for the observed phenomenon.

4. Towards a Precise Formulation of the Problem

- We want to find the optimal membrane shape.
- What does "optimal" mean?
- When the quality of an alternative can be described by a single number, optimization is straightforward.
- We select the alternative for which this number is the largest.
- In many practical cases, the situation is more complicated.
- We have several different numerical characteristics that need to be taken into account.

5. Towards a Precise Formulation (cont-d)

- What is common is all such cases is that we should be able to decide, given two alternatives a and b:
  - whether the alternative a is better; we will denote it by $b < a$,
  - or the alternative b is better: $a < b$,
  - or these two alternatives are of the same quality to the users; we will denote it by $a \sim b$.
- We can combine these two relations into a single preference relation $a \leq b$ meaning that:
  - either b is better than a
  - or b has the same quality as a.

6. Towards a Precise Formulation (cont-d)

- Once we know this combined relation:
  - we can reconstruct $a \sim b$ as $(a \le b)\,\&\,(b \le a)$, and
  - we can reconstruct $a < b$ as $(a \le b)\,\&\,(b \not\le a)$.
- Clearly, $a \le a$ for all a, i.e., the relation $\le$ must be reflexive.
- Also, if $a \le b$ and $b \le c$, then we should have $a \le c$, i.e., the relation $\le$ should be transitive.

7. Preference Relation Should Be Final

- In general, for a given preference relation:
  - we may have several different alternatives which are optimal
  - in the sense that they are better than (or of the same quality as) any other alternative.
- For example, we may have several different membranes that are all equally efficient for water desalination.
- In this case, we can use this non-uniqueness to optimize something else.
- E.g., we can select the membrane with the lowest cost or with the longest expected life.

8. Preference Relation Should Be Final (cont-d)

- So, we replace the original preference relation $\le$ with a new one $\le'$ in which $a \le' b$ if and only if:
  - either $a < b$,
  - or $a \sim b$ and $a \le_1 b$ for the additional criterion $\le_1$.
- If after that, we still have several optimal alternatives,
  - we can use this non-uniqueness
  - to optimize something else, etc.
- Finally, we get a *final* preference relation – for which there is only one optimal alternative.

9. Preference Relation Should Be Invariant with Respect to Natural Symmetries

- In many practical situations, there are natural symmetries, i.e.:
  - natural transformations
  - with respect to which the physical situation does not change.
- For example, if I drop a pen, it will fall down with the acceleration of 9.81 m/sec^2.
- If I move to another location and repeat the same experiment, I get the same result.
- In this sense, the situation does not change with shift.

10. Symmetries (cont-d)

- Similarly, if I rotate myself by 90 degrees and repeat the experiment, I get the same result.
- So, the situation is invariant with respect to rotations too.
- For the membrane, a natural transformation is shift:
  - if we move from one location of the membrane to another one, nothing should change,
  - since all the related processes are local.
- Now:
  - if there is a transformation T that does not change the physical situation,
  - then it is reasonable to require that it should not change our preference relation.

11. Symmetries (cont-d)

- So:
  - if we had $a \leq b$ for some alternatives a and b,
  - then for the transformed alternatives Ta and Tb, we should also have $Ta \leq Tb$.
- Now, we are ready to formulate the problem in precise terms.

12. Formulation of the Problem in Precise Terms

- Let A be a set. Its elements will be called *alternatives*.
- By a *preference relation* on the set A, we mean a reflexive and transitive binary relation $\leq$.
- We say that an alternative a_{opt} is *optimal* with respect to a preference relation $\leq$ if $a \leq a_{\text{opt}}$ for all $a \in A$.
- We say that a preference relation is *final* if there exists exactly one optimal alternative.
- Let $T : A \to A$ be an invertible transformation.
  - We say that an alternative a is *T-invariant* if
  $$T(a) = a.$$
  - We say that the preference relation $\leq$ is *T-invariant* if for every two alternatives a and b:
  $$a \leq b \text{ if and only if } T(a) \leq T(b).$$

13. Resulting Explanation

- **Proposition.**
  - *if the preference relation is final and T-invariant,*
  - *then the optimal alternative a_{opt} is also T-invariant.*
- In our case:
  - since the physical situation does not change with shift,
  - it is reasonable to assume that the preference relation should also be invariant with respect to shift.
- Thus, due to Proposition, we conclude that:
  - if we shift from one point to another,
  - then the optimal membrane should also not change.

14. Resulting Explanation (cont-d)

- Every two locations can be transformed into each other by an appropriate shift; thus:
  - the values of all the corresponding quantities – including density,
  - should be the same at all the locations.
- In other words, this means that the optimal membrane should be homogeneous.
- This is exactly what the experiments show.
- Thus, we have indeed showed that natural symmetry requirements explain the latest experimental results.

15. Proof

- Let $\leq$ be a final and T-invariant preference relation.
- Let a_{opt} be the alternative which is optimal with respect to this relation.
- This means that $a \leq a_{\text{opt}}$ for all $a \in A$.
- In particular, we have $T^{-1}(a) \leq a_{\text{opt}}$, where T^{-1} denotes the inverse function to $T(a)$:

$$b = T^{-1}(a) \text{ if and only if } a = T(b).$$

- Then, due to T-invariance, we conclude that $T(T^{-1}(a)) \leq T(a_{\text{opt}})$, i.e., that $a \leq T(a_{\text{opt}})$.
- This is true for all a, so, by definition of an optimal alternative, the alternative $T(a_{\text{opt}})$ is optimal.

16. Proof (cont-d)

- However, the preference relation $\leq$ is final.
- This means that there exists only one optimal alternative.
- Therefore, $T(a_{\rm opt}) = a_{\rm opt}$.
- Thus, the optimal alternative $a_{\rm opt}$ is indeed T-invariant.
- The proposition is proven.

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