Why Physical Power Laws Usually Have Rational Exponents

Edgar Daniel Rodriguez Velasquez1, 2,
Olga Kosheleva[3], Vladik Kreinovich[4]

[1]Department of Civil Engineering, Universidad de Piura in Peru (UDEP)
Av. Ramón Mugica 131, Piura, Peru, edgar.rodriguez@udep.pe
Departments of [2]Civil Engineering, [3]Teacher Education, [4]Computer Science
University of Texas at El Paso, El Paso, Texas 79968
edrodriguezvelasquez@miners.utep.edu, olgak@utep.edu, vladik@utep.edu

1. Power laws are ubiquitous

- In many application areas, we encounter power laws.
- In such laws, the dependence of a quantity y on another quantity x takes the form $y = A \cdot x^a$ for some constants A and a.

2. This ubiquity has a natural explanation

- This explanation comes from the fact that the numerical values of physical quantities depend on the choice of a measuring unit:
  - If we replace the original unit with the one which is λ times smaller,
  - then all the numerical values are *re-scaled*, namely, multiplied by λ: $x \mapsto X = \lambda \cdot x$.
- For example, 1.7 m becomes $100 \cdot 1.7 = 170$ cm.
- In many cases, there is no physically preferable measuring unit.
- In such situations, it makes sense to require that:
  - the formula $y = f(x)$ describing the dependence between x and y should be invariant (= does not change)
  - if we simply change the measuring unit for x.

3. This ubiquity has a natural explanation (cont-d)

- To be more precise:
  - for each re-scaling $x \mapsto X = \lambda \cdot x$ of the variable x,
  - there exists an appropriate re-scaling $y \mapsto Y = \mu(\lambda) \cdot Y$ of the variable Y for which $y = f(x)$ implies that $Y = f(X)$.
- Substituting the expressions for X and Y into this formula, we conclude that $f(\lambda \cdot x) = \mu(\lambda) \cdot y$.
- Since $y = f(x)$, we get $f(\lambda \cdot x) = \mu(\lambda) \cdot f(x)$.
- It is easy to check that all power laws satisfy this functional equation, for an appropriate function $\mu(\lambda)$.
- Vice versa, it is known that the only differentiable functions $f(x)$ that satisfy this functional equation are power laws.

4. Remaining problem and what we do in this paper

- Not all power laws appear in physical phenomena.
- Namely, in almost all the cases, we encounter only power laws with rational exponents a.
- How can we explain this fact?
- In this paper, we show that:
  - a natural expansion of the above invariance-based explanation for the ubiquity of power laws
  - explains why physical power laws usually have rational exponents.

5. Main idea

- The main idea behind our explanation is to take into account that for many physical quantities, their numerical values depend:
  - not only on the choice of a measuring unit,
  - but also on the choice of a starting point.
- For example:
  - if we change the starting point for measuring time to a one which is x_0 moments earlier,
  - then all numerical values of x are replaced by new "shifted" numerical values $X = x + x_0$.
- In such situations, it is reasonable to require that the formulas do not change if we simply change the starting point for x.

6. How can we apply this idea to our situation

- For a power law $y = A \cdot x^a$, we cannot apply the above idea of "shift-invariance" directly.
- If we change x to $x + x_0$, then the original power law takes a different form $y = A \cdot (x + x_0)^a$.
- This situation is somewhat similar to what we had when we derived the power law from scale-invariance:
  - if we simply replace x with $\lambda \cdot x$ in the formula $y = A \cdot x^a$,
  - we get a different formula $y = A \cdot (\lambda \cdot x)^a = (A \cdot \lambda^a) \cdot x^a$.
- So, in this sense, the power law formula $y = A \cdot x^a$ is not scale-invariant.
- What *is* scale-invariant is the 1-parametric family of functions $\{C \cdot (A \cdot x^a)\}_C$ corresponding to different values C.
- This scale-invariance is exactly what leads to the power law.

7. How can we apply this idea to our situation (cont-d)

- With respect to shifts, however, the 1-parametric family $\{C\cdot(A\cdot x^a)\}_C$ is *not* invariant.
- So, a natural idea is to consider a multi-parametric family, i.e.:
  - to fix several functions $e_1(x), \ldots, e_n(x)$, and
  - to consider the family of all possible linear combinations of these functions:
    $$\{C_1 \cdot e_1(x) + \ldots + C_n \cdot e_n(x)\}_{C_1,\ldots,C_n}.$$
- For this family, we can try to require both scale- and shift-invariance.
- In this case, we get the following result.

8. How can we apply this idea to our situation (cont-d)

- **Proposition.** *Let* $e_1(x), \ldots, e_n(x)$ *be differentiable functions for which:*
  - *the family of their linear combinations is scale- and shift-invariant and*
  - *for which this family contains a power law* $f(x) = A \cdot x^a$*.*

  Then, a *is an integer.*

9. How can we apply this idea to our situation (cont-d)

- **Proof of the Proposition.** It is known that:
  - if for some differentiable functions $e_i(x)$, the family of linear combinations is scale- and shift-invariant,
  - then all the functions from this family are polynomials, i.e., functions of the type $a_0 + a_1 \cdot x + \ldots + a_p \cdot x^p$.
- In particular, this means that the power law $f(x) = A \cdot x^a$ – which is also a member of this family – is a polynomial.
- The only case when the power law is a polynomial is when the exponent a is a non-negative integer.
- The proposition is proven.

10. How this explains the prevalence of rational exponents

- What we proved so far was an explanation of why we often have integer exponents $y = A \cdot x^n$ for some integer n.
- But how can we get rational exponents?
- First, we notice that:
  - if the dependence of y on x has the form $y = A \cdot x^n$, with an integer exponent n,
  - then the dependence on x on y has the form $x = B \cdot x^{1/n}$ for some constant B, with a rational exponent which is not an integer.
- Another thing to notice is that the relation between two quantities x and y is rarely direct.
- For example, it may be that y depends on some auxiliary quantity z which, in turn, depends on x.

11. How this explains the prevalence of rational exponents (cont-d)

- In general:
  - y depends on some auxiliary quantity z_1,
  - this quantity depends on another auxiliary quantity z_2, etc., and
  - finally, the last auxiliary quantity z_k depends on x.
- If all these dependencies are described by power laws, then we have

$$y = A_0 \cdot z_1^{a_0}, \quad z_1 = A_1 \cdot z_2^{a_1}, \ldots, z_{k-1} = A_{k-1} \cdot z_k^{a_{k-1}}, z_k = A_k \cdot x^{a_k}.$$

- Here, the coefficients a_i which are either integers or inverse integers.
- Then, we have

$$z_{k-1} = A_{k-1} \cdot z_k^{a_{k-1}} = A_{k-1} \cdot (A_k \cdot x^{a_k})^{a_{k-1}} = \text{const} \cdot x^{a_{k-1} \cdot a_k},$$

$$z_{k-2} = A_{k-2} \cdot z_{k-1}^{a_{k-2}} = A_{k-2} \cdot (\text{const} \cdot x^{a_{k-1} \cdot a_k})^{a_{k-2}} = \text{const} \cdot x^{a_{k-2} \cdot a_{k-1} \cdot a_k}, \ldots$$

- Finally $y = \text{const} \cdot x^a$, where $a = a_0 \cdot a_1 \cdot \ldots \cdot a_k$.

12. How this explains the prevalence of rational exponents (cont-d)

- Since all the values a_i are rational numbers, their product is also rational.
- Every rational exponent $a = m/n$ can be thus obtained, if we take

$$y = \text{const} \cdot z_1^m \text{ and } z_1 = \text{const} \cdot x^{1/n}.$$

- This explains why rational exponents are ubiquitous.

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