How People Make Decisions Based on Prior Experience: Formulas of Instance-Based Learning Theory (IBLT) Follow from Scale Invariance

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1. How do people make decisions?

- To properly make a decision, i.e., to select one of the possible actions, we need to predict the consequences of each of these actions.
- To predict the consequences of each action, we take into account past experience, in which we know the consequences of similar actions.
- Often, at different occasions, the same action led to different consequences.
- So, we cannot predict what exactly will be the consequence of each action.
- At best, for each action, we can try to predict the probability of different consequences.
- In this prediction, we take into account the frequency with which each consequence occurred in the past.

2. How do people make decisions (cont-d)

- We also take into account that situations change.
- So more recent observations should be given more weight than the ones that happened long ago.
- To better understand human behavior, we need to know how people take all this into account.

3. Semi-empirical formulas

- Researchers performed experiments and analyzed the resulting data.
- They found semi-empirical formulas that provide a very good description of the actual human behavior.
- These formulas form the basis of the Instance-Based Learning Theory (IBLT).
- In the first approximation, when we only consider completely different consequences, these formulas have the following form.
- For each action, to estimate the probability p_i of each consequence i, we first estimate the activation A_i of this consequence as

$$A_i = \ln\left(\sum_j (t - t_{i,j})^{-d}\right).$$

4. Semi-empirical formulas (cont-d)

- Here:
  - t is the current moment of time (i.e., the moment of time at which we make a decision),
  - the values $t_{i,1}$, $t_{i,2}$, etc. are past moments of time at which the same action led to consequence i, and
  - $d > 0$ is a constant – depending on the decision maker.
- Based on these activation values, we estimate the probability p_i as

$$p_i = \frac{\exp(c \cdot A_i)}{\sum\limits_{k} \exp(c \cdot A_k)}.$$

- Here c is another constant depending on the decision maker, and the summation in the denominator is over all possible consequences k.

5. Semi-empirical formulas (cont-d)

- How can we explain why these complex formulas properly describe human behavior?
- In this paper, we show that these formulas can be actually derived from the natural idea of scale invariance.

6. Analysis of the problem

- If time was not the issue, then:
  - the natural way to compare different consequences i
  - would be by comparing the number of times n_i that the i-th consequence occurred in the past: $n_i = \sum_j 1$.
- In this formula, for each past observation of the i-th consequence, we simply add 1.
- In other words, all observations are assigned the same weight.
- As we have mentioned, it makes sense to provide larger weight to more recent observations and smaller weight to less recent ones.
- In other words, instead of adding 1s, we should add some weights depending on the time $\Delta t = t - t_{i,j}$ that elapsed since the observation.
- Let us denote the dependence on the weight on time by $f(\Delta t)$.

7. Analysis of the problem (cont-d)

- The function $f(\Delta t)$ should be decreasing with Δt: the more time elapsed, the smaller the weight.
- In these terms, the above simplified formula should be replaced by the following more adequate formula $n_i = \sum\limits_j f(t - t_{i,j})$.
- The corresponding values n_i describe the time-adjusted number of observations.
- Based on the values n_i, we need to predict the corresponding probabilities p_i.
- The more frequent the consequence, the higher should be its probability.

8. Analysis of the problem (cont-d)

- At first glance, it may seem that we can simply take $p_i = g(n_i)$ for some increasing function $g(z)$.
- However, this will not work, since the sum of the probabilities of different consequences should be equal to 1.
- To make sure that this sum is indeed 1, we need to "normalize" the values $g(n_i)$, i.e., to divide each of them by their sum:

$$p_i = \frac{g(n_i)}{\sum\limits_k g(n_k)}.$$

- The above formulas leave us with a natural question: which functions $f(t)$ and $g(z)$ better describe human behavior?

9. Our main idea: scale-invariance

- To answer this question, let us take into account that:
  - the numerical value of time duration – as well as the numerical values of many other physical quantities
  - depends on the choice of the measuring unit.
- Namely:
  - if we replace the original unit for measuring time by a new unit which is $\lambda > 0$ times smaller,
  - then all numerical values of time intervals get multiplied by λ: $t \mapsto \lambda \cdot t$.
- For example, if we replace minutes by seconds, then all numerical values are multiplied by 60.
- So, e.g., 2 minutes becomes 120 seconds.

10. Our main idea: scale-invariance (cont-d)

- In many physical (and other) situations, there is no physically preferred unit for measuring time intervals x.
- This means that the formulas should remain the same:
  - if we "re-scale" t by choosing a different measuring unit,
  - i.e., if we replace all numerical values t with $t' = \lambda \cdot t$.
- This "remains the same" is called scale invariance.
- Now, we are ready to formulate our main result.

11. Main result

Let $f(x)$ and $g(x)$ by continuous monotonic functions. Then, the following conditions are equivalent to each other:

- *for each $\lambda > 0$, the values of p_i as described by the below formulas will remain the same if we replace t and $t_{i,j}$ with $t' = \lambda \cdot t$ and $t'_{i,j} = \lambda \cdot t_{i,j}$:*

$$n_i = \sum_j f(t - t_{i,j}), \quad p_i = \frac{g(n_i)}{\sum_k g(n_k)}.$$

- *the dependence is described by the IBLT formulas:*

$$A_i = \ln\left(\sum_j (t - t_{i,j})^{-d}\right); \quad p_i = \frac{\exp(c \cdot A_i)}{\sum_k \exp(c \cdot A_k)}.$$

12. Proof

- Let us first show that if we re-scale all the time values in IBLT formulas, the probabilities remain the same.
- Indeed, in this case,
$$\sum_j (t' - t'_{i,j})^{-d} = \lambda^{-d} \cdot \sum_j (t - t_{i,j})^{-d}.$$
- Thus, the new value A'_i of the activity is:
$$A'_i = \ln\left(\sum_j (t' - t'_{i,j})^{-d}\right) = \ln\left(\sum_j (t - t_{i,j})^{-d}\right) + \ln(\lambda^{-d}) = A_i + \ln(\lambda^{-d}).$$
- Thus, we have $c \cdot A'_i = c \cdot A_i + c \cdot \ln(\lambda^{-d})$, and $\exp(c \cdot A'_i) = C \cdot \exp(c \cdot A_i)$.
- Here we denoted $C \stackrel{\text{def}}{=} \exp(c \cdot \ln(\lambda^{-d}))$.
- Hence, the new expression for probability takes the form
$$p'_i = \frac{\exp(c \cdot A'_i)}{\sum\limits_k \exp(c \cdot A'_k)} = \frac{C \cdot \exp(c \cdot A_i)}{C \cdot \sum\limits_k \exp(c \cdot A_k)}.$$

13. Proof (cont-d)

- Let us divide both the numerator and the denominator of the right-hand side by the same constant C.
- Then, we can conclude that $p'_i = p_i$, i.e., that the probabilities indeed do not change.
- So, to complete our proof, it is sufficient to prove that:
  - if the general expressions lead to scale-invariant probabilities,
  - then the dependence is described by IBLT formulas.
- Let us first find what we can deduce from scale-invariance about the function $g(x)$.
- To do that, let us consider the case when we have two consequences:
  - one of which was observed only once, and
  - the other one was observed m times for some $m > 1$.
- Let us also assume that all the observations occurred at the same time Δt moments in the past, so that $t - t_{i,j} = \Delta t$.

14. Proof (cont-d)

- In this case, $n_1 = f(\Delta t)$, $n_2 = m \cdot n_1$, and the formula for the probability p_1 takes the form $p_1 = \dfrac{g(f(\Delta t)}{g(m \cdot f(\Delta t)) + g(f(\Delta t))}$.
- By re-scaling time, we can replace Δt with any other value, and this should not change the probabilities.
- Thus, the above formula for p_1 should retain the same value for all possible values of $z = f(\Delta t)$.
- In other words, the ratio $\dfrac{g(z)}{g(m \cdot z) + g(z)}$ should not depend on z, it should only depend on m.
- Hence, its inverse $\dfrac{g(m \cdot z) + g(z)}{g(z)} = \dfrac{g(m \cdot z)}{g(z)} + 1$ should also depend only on m.
- Therefore, we should have $\dfrac{g(m \cdot z)}{g(z)} = a(m)$ for some function $a(m)$.

15. Proof (cont-d)

- So, we should have $g(m \cdot z) = a(m) \cdot g(z)$ for all z and m.
- In particular, for $z' = z/m'$ for which $m' \cdot z' = z$, we should have $g(m' \cdot z') = a(m') \cdot g(z')$, i.e., $g(z) = a(m') \cdot g(z/n')$, hence
$$g(z/n') = (1/a(m')) \cdot g(z).$$
- So, for each rational number $r = m/m'$, we should have
$$g(r \cdot z) = g(m \cdot (z/m')) = a(m) \cdot g(z/m') = a(m) \cdot (1/a(m')) \cdot g(z).$$
- In other words, for every z and for every rational number r, we should have $g(r \cdot z) = a(r) \cdot g(z)$, where we denoted $a(r) \stackrel{\text{def}}{=} a(m) \cdot (1/a(m'))$.
- By continuity, we can conclude that the formula $g(r \cdot z) = a(r) \cdot g(z)$ holds for all real values r, not necessarily for rational values.
- It is known that every continuous solution to this functional equation is the power law, i.e., $y = A \cdot x^a$ for some constants A and a.
- Thus, we conclude that $g(z) = A \cdot z^a$.

16. Proof (cont-d)

- We can simplify the resulting expression for $g(z)$ even more.
- Indeed, substituting this expression into the formula for p_i, we conclude that $p_i = \frac{A \cdot n_i^a}{A \cdot \sum\limits_k n_k^a}$.
- We can simplify this expression if we divide both the numerator and the denominator by the same constant A.
- Then, we get the following simplified formula $p_i = \frac{n_i^a}{\sum\limits_k n_k^a}$.
- This formula corresponds to the function $g(z) = z^a$.
- So, without loss of generality, we can conclude that $g(z) = z^a$.

17. Proof (cont-d)

- Let us now find out what we can deduce from scale-invariance about the function $f(t)$.
- For this purpose, let us consider two consequences each of which was observed exactly once:
  - one of which was observed 1 time unit ago and
  - the other one was observed t_0 time units ago.
- Then, according the general formulas, the predicted probability p_1 should be equal to $p_1 = \dfrac{(f(1))^a}{(f(1))^a + (f(t_0))^a}$.
- By scale-invariance, this probability should not change if we multiply both time intervals by λ, so that $1 \mapsto \lambda$ and $t_0 \mapsto \lambda \cdot t_0$:

$$\frac{(f(1))^a}{(f(1))^a + (f(t_0))^a} = \frac{(f(\lambda))^a}{(f(\lambda))^a + (f(\lambda \cdot t_0))^a}.$$

18. Proof (cont-d)

- The equality remains valid if we take the inverses of both sides:
$$\frac{(f(1))^a + (f(t_0))^a}{(f(1))^a} = \frac{(f(\lambda))^a + (f(\lambda \cdot t_0))^a}{(f(\lambda))^a}.$$
- Let us subtract 1 from both sides: $\dfrac{(f(t_0))^a}{(f(1))^a} = \dfrac{(f(\lambda \cdot t_0))^a}{(f(\lambda))^a}$ and raise both sides to the power $1/a$:
$$\frac{f(t_0)}{f(1)} = \frac{f(\lambda \cdot t_0)}{f(\lambda)}.$$
- The left-hand side of this equality does not depend on λ, it depends only on t_0.
- Thus, the right-hand side should also depend only on t_0, i.e., we should have $\dfrac{f(\lambda \cdot t_0)}{f(\lambda)} = F(t_0)$, for some function $F(t_0)$.
- Multiplying both sides by $f(\lambda)$, we conclude that $f(\lambda \cdot t_0) = F(t_0) \cdot f(\lambda)$ for all $t_0 > 0$ and $\lambda > 0$.

19. Proof (cont-d)

- We have already mentioned that every continuous solution to this functional equation has the form $f(t) = B \cdot t^b$ for some B and b.
- Since the function $f(t)$ is decreasing, we have $b < 0$.
- We can simplify the expression for $f(t)$ even more.
- Indeed, substituting the expression for $f(t)$ into the formula for n_i, we get $n_i = B \cdot \sum\limits_j (t - t_{i,j})^b$, i.e., $n_i = B \cdot a_i$, where we denoted

$$a_i \stackrel{\text{def}}{=} \sum_k (t - t_{i,j})^b.$$

- Substituting the formula $n_i = B \cdot a_i$ into the formula for p_i, we get

$$p_i = \frac{B^a \cdot a_i^a}{B^a \cdot \sum\limits_k a_k^a}.$$

- We can simplify this expression if we divide both the numerator and the denominator by the same constant B^a.

20. Proof (cont-d)

- Then, we get the following simplified formula $p_i = \dfrac{a_i^a}{\sum\limits_k a_k^a}$.
- This formula corresponds to using a_i instead of n_i, i.e., in effect, to using the function $f(t) = t^b$.
- So, without loss of generality, we can conclude that $f(t) = t^b$.
- For this function $f(t)$, we have $p_i = \dfrac{n_i^a}{\sum\limits_k n_k^a}$.
- Let us show that for $f(t) = t^b$ for $b < 0$ and $g(z) = z^a$, we indeed get the IBLT expression.
- Indeed, for $f(t) = t^b$, the formula for n_i takes the form
$$n_i = \sum_j (t - t_{i,j})^b.$$
- So, it has the form $n_i = \sum_j (t - t_{i,j})^{-d}$, where we denoted $d \stackrel{\text{def}}{=} -b$.

21. Proof (cont-d)

- Thus, we get $A_i = \ln(n_i)$.
- So, the expression $\exp(c \cdot A_i)$ in the IBLT empirical formula takes the form
$$\exp(c \cdot A_i) = \exp(c \cdot \ln(n_i)) = (\exp(\ln(n_i))^c = n_i^c.$$
- Thus, the IBLT formula takes the form $p_i = \dfrac{n_i^c}{\sum\limits_k n_k^c}$.
- One can see that this is exactly our formula.
- The only difference is that the parameters that is denoted by c in the IBLT formula is denoted a in our formula.
- Thus, we have indeed explained the empirical IBLT formulas.
- The proposition is proven.

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