Conflict Situations Are Inevitable When There Are Many Participants: A Proof Based on the Analysis of Aumann-Shapley Value

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1. Collaboration is often beneficial

- In many practical tasks, be it menial or intellectual tasks, it is beneficial for several people to collaborate.

- This way:
  - every participant is focusing on the task in which he/she is most skilled
  - while tasks at which this participant is not very skilled are performed by those who are better in these tasks.

- In such situations, in general, the more people participate, the better the result.

2. Question: how to divide the resulting gain

- Often, the resulting gain is financial:

  - the company gets a profit,

  - a research group gets a bonus or an award, etc.

- A natural question is: what is the fair way to divide this gain between the participants.

3. Shapley value: a description of a fair division

- There is a known answer to this question.

- This answer was originally produced by the Nobelist Lloyd Shapley.

- Let us describe this answer.

- Let n denote the number of collaborators, and let us number them by numbers from 1 to n.

- To describe the contribution of each participant:

  - for each subset $S \subseteq N \overset{\text{def}}{=} \{1, \ldots, n\}$,
  - we can estimate the gain $v(S)$ that participants from the set S would have gained without any help from the others.

- Based on the function $v(S)$, we need to determine how to divide the overall gain $v(N)$ between the participants.

- We need to come up with the values $x_i(v)$ for which

$$x_1(v) + \ldots + x_n(v) = v(N).$$

4. Shapley value: a description of a fair division (cont-d)

- Shapley introduced natural requirements on the function $x_i(v)$.

- First, this function should not depend on the numbers that we assign to the participants.

- If we start with a different participant etc., each participant should receive the same portion as before.

- Second, people may participate in two different collaborative projects, corresponding to functions $u(S)$ and $v(S)$.

- As a result of the first project, each participant i gets $x_i(u)$.

- As the result of the second project, this participant gets $x_i(v)$.

- Thus, the overall amount gained by the i-th participant is

$$x_i(u) + x_i(v).$$

- Alternatively, we can view the two projects as two parts of one big project.

5. Shapley value: a description of a fair division (cont-d)

- In this case, for each set S, the gain $w(S)$ is equal to the sum of their gains in the two parts: $w(S) = u(S) + v(S)$.

- Based on this overall project, the i-th participant should get the value

$$x_i(w) = x_i(u + v).$$

- It is reasonable to require that the portion assign to the i-th participant should not depend on:

  - whether we treat two projects separately or
  - whether we treat them as two parts of a big project.

- In other words, we should have $x_i(u + v) = x_i(u) + x_i(v)$.

- This property is known as *additivity*.

6. Shapley value: a description of a fair division (cont-d)

- Shapley has shown that these two natural requirements uniquely determine the function $x_i(v)$: namely, we should have

$$x_i(v) = \sum_{S:\ i \notin S} \frac{|S|! \cdot (n - |S| - 1)!}{n!} \cdot (v(S \cup \{i\}) - v(S)).$$

- Here $|S|$ denotes the number of elements in the set S. This formula is known as *Shapley value*.

7.   Continuous approximation: idea

- As we have mentioned, for large number of participants, using the above formula to compute the Shapley value is not feasible.

- Such situations – when a large number of objects makes computations difficult – are common in science.

- For example, we know that a solid body consists of molecules, and we know the equations that describe how the molecules interact.

- However, for a large number of molecules forming a body it is not realistic to take all the molecules into account.

- Instead, we use a continuous approximation:
  - we assume that the matter is uniformly distributed, and
  - we perform computations based on this assumption.

8. Continuous approximation: towards a precise formulation of the problem

- In the continuous approximation, the set of participants forms an area A in 1-D or 2-D or multi-dimensional space.

- For example:
    - if we want to consider the best way to divide the budget surplus in the state of Texas between projects benefiting local communities,
    - we can view the set of participation as the 2-D area of the state.

- If we want to treat people with different income level differently, we should consider the set of participants as a 3-D region:
    - the first two parameters are geographic coordinates, and
    - the third parameter is the income.

- If we take more parameters into account, we get an area of larger dimension.

9. Continuous approximation: towards a precise formulation of the problem (cont-d)

- As we mentioned earlier, to describe the situation, we assign:

    - to each subset S of the original set of participants,
    - the value $v(S)$ describing how much participants from this set can gain without collaboration with others.

- In the continuous approximation, we consider reasonable (e.g., measurable) subsets S of the original area A.

- We also consider reasonable functions $v(S)$, e.g., functions of the type $F(v_1(S), \ldots, v_k(S))$, where:

    - $F(z_1, \ldots, z_k)$ is a continuous function, and
    - each v_i has the form $v_i(S) = \int_S f_i(x) \, dx$ for some bounded measurable function $f_i(x)$.

10. Continuous approximation: towards a precise formulation of the problem (cont-d)

- Let us denote the set of all such functions by V.

- This set V is closed under addition and under a multiplication by a positive number, i.e.:

  - if $u, v \in V$ and $\alpha > 0$,
  - then $u + v \in V$ and $\alpha \cdot v \in V$.

11. Continuous approximation: towards a natural solution

- In the original formulation:

  - based on the known values $v(S)$ corresponding to different sets $S \subseteq N$,

  - we decide what portion $x_i(v)$ of the gain $v(N)$ to allocated to each participant i.

- Once we decide on this, to each group $S \subseteq N$, we thus allocate the sum $x_S(v)$ of values x_i allocated to all the members of this group:

$$x_S(v) = \sum_{i \in S} x_i(v).$$

- How can we extend this formula to the continuous case?

- To do this, we can use the experience of physicists who use the continuous approximation to predict the properties of a solid body.

12. Continuous approximation: towards a natural solution (cont-d)

- Their technique is:

  - to discretize the space, i.e., to divide the area occupied by the solid body into small cells,

  - to approximate the behavior of each physical quantity inside a cell by a few parameters (e.g., by a constant), and

  - to solve the corresponding finite-parametric problem.

- In other words, the usual idea is to provide a discrete approximation to the continuous approximation.

- The Nobelists Aumann and Shapley proposed to apply the same idea to the continuous approximation to the gain-dividing problem.

13. Continuous approximation: towards a natural solution (cont-d)

- To approximate the continuous game with a sequence of discrete games, they proposed to consider:

    - sequences of partitions $P^{(1)}, P^{(2)}, \ldots, P^{(k)}, \ldots$
    - each of which divides the area A into a finite number of disjoint measurable sub-areas $A_1^{(k)}, \ldots, A_{n_k}^{(k)}$.

- Each sequence should satisfy the following two properties:

    - the next division $P^{(k+1)}$ is obtained from the previous division $P^{(k)}$ by sub-dividing each of the sets $A_i^{(k)}$, and
    - for every $a \neq b$, there exists k for which $P^{(k)}$ allocates a and b to different sub-areas.

- Such sequences of partitions are called *admissible*.

14. Continuous approximation: towards a natural solution (cont-d)

- For each partition $P^{(k)}$ for an admissible sequence:

  - we consider sets S consisting of the corresponding sub-areas,
  - i.e., sets S of the type $S(s) = \bigcup_{i \in s} A_i^{(k)}$ for some set

$$s \subseteq \{1, \ldots, n_k\}.$$

- In this discrete-approximation-to-continuous-approximation scheme, we get, for each k, a situation with n_k participants.

- In this situation, the gain of each subset $s \subseteq \{1, \ldots, n_k\}$ is described by the value $v(S(s))$.

- Based on these values, we can compute, for each of these participants, the Shapley value $x_i^{(k)}(v)$.

15. Continuous approximation: towards a natural solution (cont-d)

- For each measurable subset $S \subseteq A$ and for each partition $P^{(k)}$, we get the approximate lower bound for the amount allocated to S:

$$x_S^{(k)}(v) = \sum_{i:\ A_i^{(k)} \subseteq S} x_i^{(k)}(v).$$

- Some functions $v(S)$ have the following nice property:

  - for each set S, no matter what admissible sequence of partitions $P^{(k)}$ we take,
  - the values $x_S^{(k)}(v)$ tend the same limit.

- This limit $x_S(v)$ is called the *Aumann-Shapley value* corresponding to the function $v(S)$.

16. Sometimes, the Aumann-Shapley value exists, sometimes, it does not exist

- For some functions $v(S)$, the sequences $x_S^{(k)}(v)$ always converge; so:
  - if the original situation with a large number of participants can be described by such function $v(S)$,
  - then, to fairly allocate gains to all the participants, we can use an appropriate simplified approximation.

- On the other hand, it is known that for some functions $v(S)$, the sequences $x_S^{(k)}(v)$ do not converge.

- For example, the limits do not exist already for the following simple function on the domain $A = [0, 1]$:

$$v(S) = \min\left(\int_S \chi_{\left[0, \frac{1}{3}\right)} \, dx, \int_S \chi_{\left[\frac{1}{3}, \frac{2}{3}\right)} \, dx, \int_S \chi_{\left[\frac{2}{3}, 1\right]} \, dx\right).$$

- Here χ_I is the characteristic function of the set I, i.e., $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ is $x \notin I$.

- For such functions v, we cannot use a simplified approximation.

- For large number of participants, direct computation of the Shapley value – i.e., of the fair division – is not feasible.

- This means that for such functions $v(S)$, we cannot feasibly compute a division that everyone would recognize as fair.

- Thus, in such situations, conflicts are inevitable.

18. Natural question

- A natural question is: which of these two situations occurs in real life?

- In this talk, we come up with an answer to this question.

19.   Main idea behind our analysis

- To get an answer to the above question, we will use the following principle that physicists use:

  - if some phenomenon occurs in almost all situations ("almost all" in some reasonable sense),

  - then they conclude that this phenomenon occurs in real life as well.

- For example, suppose that you flip a coin many times and select 1 when it falls head and 0 when it falls tail.

- In principle, you can get any sequence of 0s and 1s.

- However, we know that for almost all sequences of 0s and 1s, the frequency of 1s tends to $1/2$.

- So, physicists conclude (and experiments confirm this) that when we flip a coin many times, the frequency of 1s tends to $1/2$.

20.   Main idea behind our analysis (cont-d)

- This is not just what physicists do, this is common sense:

  - if you go to a casino and the roulette ends up on red (as opposed to black) 30 times in a row,

  - you will naturally conclude that the roulette is biased.

- Similarly: it is, in principle, possible that:

  - due to random thermal interactions between all the molecules in a cat's body,

  - all the molecules will start moving up, and the poor cat will start rising in the air.

- However, the probability of this event is practically 0.

- In almost all cases, this is not possible, so physicists conclude that this is not possible in real life.

21. Main idea behind our analysis (cont-d)

- An even simpler example:

  - if we run some stochastic process twice and we get the exact same result in both cases,

  - this would mean that something is wrong.

- Indeed, once we have the first result r_1, the second result r_2 can, in principle, take any real value.

- Only for one of these values r_1, out of continuum many, we can have

$$r_2 = r_1.$$

- Thus, for almost all possible values r_2 (with one exception), we have

$$r_2 \neq r_1.$$

- Thus, we conclude that in real life, we will have $r_2 \neq r_1$.

22. In our analysis, we will take into account additivity and homogeneity of the Aumann-Shapley value

- It is known that several properties of the Shapley value can be extended to the Aumann-Shapley value.

- We just need to take into account that:

  - in contrast to the Shapley value – which is always defined,
  - the Aumann-Shapley value is only defined for some functions $v(S)$.

- As we have mentioned, the Shapley value has the additivity property: for every two functions $u(S)$ and $v(S)$, we have

$$x_i(u + v) = x_i(u) + x_i(v).$$

- Also, it has the following homogeneity property: for every $\alpha > 0$, we have $x_i(\alpha \cdot v) = \alpha \cdot x_i(v)$.

- In the limit, these two properties leads to the following properties of the Aumann-Shapley (AS) value.

- If the AS value exists for functions u and v, then AS value exists for $u + v$, and $x_S(u + v) = x_S(u) + x_S(v)$.

- If the AS value exists for functions u and $u + v$, then AS value exists for v, and $x_S(u + v) = x_S(u) + x_S(v)$.

- If the AS value exists for v, then for each $\alpha > 0$, the AS value exists for $\alpha \cdot v$, and $x_S(\alpha \cdot v) = \alpha \cdot x_S(v)$.

24. What we can conclude from these properties

- Let v_0 be a function for which the AS value does not exist.

- For each function $v \in V$, we can consider a 1-D set

$$L(v) \overset{\text{def}}{=} \{v + t \cdot v_0 \,|\, v + t \cdot v_0 \in V\}.$$

- Here t denotes an arbitrary real number.

- For all $t \geq 0$, we have $v + t \cdot v_0 \in V$, so this set contains a whole half-line, i.e., it contains continuum many points.

- Let us prove that in this set, we can have at most one real value t for which the AS value exist.

- The proof is by contradiction.

- Indeed, suppose that we have two values $t_1 < t_2$ for which the functions $v + t_i \cdot v_0$ have AS value.

- Then, by additivity, the AS value exists for the difference

$$(v + t_2 \cdot v_0) - (v + t_1 \cdot v_0) = (t_2 - t_1) \cdot v_0.$$

- Thus, by homogeneity, the AS value exists for the function

$$(t_2 - t_1)^{-1} \cdot ((t_2 - t_1) \cdot v_0) = v_0.$$

- However, we have selected v_0 for which the AS value does not exist.

- The resulting contradiction proves our statement.

26. This means that for almost all functions v, the AS value does not exist

- One can easily check that for two functions v and v':

  - if the corresponding sets $L(v)$ and $L(v')$ have a common element,
  - then these sets coincide.

- Thus, we can divide the whole set V of all possible functions $v(S)$ into non-intersecting 1-D subsets $L(v)$ corresponding to different v.

- In each such subset, out of continuum many points, there is at most one point for which the AS value exists;

  - when some property holds only for one point on the whole half-line,
  - this means, intuitively, that in almost all cases, this property is not satisfied.

- This is exactly what we wanted to prove.

27.   Corollary: reminder

- We have shown that for almost all functions $v(S)$:

  - the AS value does not exist,
  - i.e., that the Shapley values corresponding to discrete few-player approximations do not converge.

- Following the above physicists' principle, we conclude that in real life, these values do not converge.

- Thus, in situations with many participants, we cannot use this approximation idea.

- For large number of participants, direct computation of the Shapley value (i.e., of the fair division) is not feasible.

- This means that for such functions $v(S)$, we cannot feasibly compute a division that everyone would recognize as fair.

- Thus, in such situations, conflicts are inevitable.

28.   Acknowledgments

29. References

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