What Do Goedel's Theorem and Arrow's Theorem Have in Common: A Possible Answer to Arrow's Question

Miroslav Svítek[1], Olga Kosheleva[2], and Vladik Kreinovich[2]
[1]Faculty of Transportation Sciences, Czech Technical University in Prague,
Konviktska 20, CZ-110 00 Prague 1, Czech Republic, svitek@fd.cvut.cz
University of Texas at El Paso, El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu

1. Formulation of the problem

- Kenneth Arrow is the renowned author of the Impossibility Theorem that explains the difficulty of group decision making.
- He noticed that there is some commonsense similarity between:
  - his result and
  - Goedel's theorem about incompleteness of axiomatic systems.
- Arrow asked if it is possible to describe this similarity in more precise terms.
- In this paper, we make the first step towards this description.
- We show that in both cases, the impossibility result disappears if we take into account probabilities.

2. Formulation of the problem (cont-d)

- Namely, we take into account:
  - we can consider probabilistic situations,
  - that we can make probabilistic conclusions, and
  - that we can make probabilistic decisions (when we select different alternatives with different probabilities).

3. Need to consider the axiomatic approach

- Both Goedel's and Arrow's results are based on axioms.
- So, to analyze possible similarities between these two results, let us recall where the axiomatic approach came from.
- Already ancient people had to deal the measurements of lengths, angles, areas, and volumes.
- This was important in constructing building.
- This was important for deciding how to re-mark borders between farms after a flood, etc.
- The experience of such geometric measurements led to the discovery of many interesting relations between all these quantities.

4. Need to consider the axiomatic approach (cont-d)

- For example, people empirically discovered what we now call Pythagoras theorem – that:
  - in a right triangle,
  - the square of the hypothenuse is equal to the sum of the square of the sides.
- At first, this was just a collection of interesting and useful relations.
- After a while, people noticed that some of these relations can be logically deduced from others.
- Eventually, it turned out that it is enough to state a small number of these relations.
- These selected relations became known as *axioms*.
- Every other geometric relation, every other geometric fact, every other statement about geometric objects can be deduced from these axioms.

5. Need to consider the axiomatic approach (cont-d)

- In general, it was believed that for each geometric statement, we can deduce, from the axioms of geometry:
  - either deduce this statement
  - or its negation.

6. Can axiomatic method be applied to arithmetic?

- The success of axiomatic method in geometry led to a natural idea of using this method in other disciplines as well.
- Geometry is part of mathematics.
- So, a natural idea is to apply axiomatic method to other parts of mathematics, starting with statements about natural numbers.
- The corresponding axioms have indeed been formulated in the 19th century.
- Most mathematicians believed that – similarly to geometry:
  - for each statement about natural numbers,
  - we can either deduce this statement or its negation from these axioms.

7. Can axiomatic method be applied to arithmetic (cont-d)

- OK, some mathematicians were not 100% sure that the available axioms would be sufficient for this purpose.
- However, they were sure that in this case, we can achieve a similar result if we add a few additional axioms.

8. Goedel's impossibility result

- Surprisingly, it turned out that the desired complete description of natural numbers is not possible.
- No matter what axioms we select:
  - either there is a statement for which neither this statement not its negation can be derived from the axioms,
  - or the set of axioms is inconsistent – i.e., for each statement, we can deduce *both* this statement and its negation from these axioms.
- In other words:
  - it is not possible to formulate the set of axioms about natural numbers
  - that would satisfy the natural condition of completeness.
- This result was proven by Kurt Goedel.

9. Goedel's impossibility result (cont-d)

- In particular, Goedel proved that:
  - the incompleteness can already be shown
  - for statements of the type $\forall n\, P(n)$ for algorithmically decidable formulas $P(n)$.

10. Arrow's impossibility result

- There is another area of research where we have reasonable requirements: namely, human behavior.
- Already Baruch Spinoza tried to describe human behavior in axiomatic terms.
- In the 20th century, researchers continued such attempts.
- In particular, attempts were made to come up with a scheme of decision making that would satisfy natural fairness restrictions.
- Similar to natural numbers, at first, researchers hoped that such a scheme would be possible.

11. Arrow's impossibility result (cont-d)

- However, in 1951, the future Nobelist Kenneth Arrow proved his famous Impossibility Theory:
  - it is not possible to have an algorithm that, given preferences of several participants
  - would come up with a group decision that would satisfy natural fairness requirements.
- For this result, Arrow was awarded the Nobel prize.

12. What do Goedel's and Arrow's impossibility theorems have in common: Arrow's question

- From the commonsense viewpoint, these two results are somewhat similar.
- They are both about impossibility of satisfying seemingly reasonable conditions.
- However, from the mathematical viewpoint, these two results are very different:
  - the formulations are different,
  - the proofs are different, etc.
- Arrow conjectured that there must be some mathematical similarity between these two results as well.

13. What we do in this paper

- In this paper, we show that, yes, there is some mathematical similarity between these two results.
- Our result just scratches the surface.
- However, we hope that it will lead to discovering deeper and more meaningful similarities.

14. Implicit assumption underlying Arrow's result

- Arrow's result implicitly assumed that all our decisions are deterministic.
- But is this assumption always true?
- The fact that this assumption is somewhat naive can be best illustrated by the known argument about a donkey.
- This example was first described by a philosopher Buridan.
- According to this argument, a donkey placed between two identical heaps of hay:
  - will not be able to select one of them and
  - will, thus, die of hunger.
- Of course, the real donkey will not die.
- It will select one of the heaps at random and start eating the juicy hay.

15. Implicit assumption underlying Arrow's result (cont-d)

- Similarly, a human:
  - when facing a fork in a road – without any information about possible paths,
  - will randomly select one of the two directions.
- When two friends meet for dinner and each prefers his own favorite restaurant, they will probably flip a coin to decide where to eat.
- In other words, people not only make deterministic decisions.
- They sometimes make probabilistic decisions, i.e., they select different actions with different probabilities.

16. Let us take probabilities into account when describing preferences and decisions

- Since probabilistic decisions are possible, we need to take them into account – both:
  - when we describe preferences and
  - when we select a decision.
- This taking into account leads to so-called *decision theory*.
- We will show that it helps to resolve the paradox of Arrow's theorem.
- Namely, it leads to a reasonable way to select a joint decision.
- From this viewpoint, Arrow's theorem means that:
  - if we only know preferences between *deterministic* alternatives,
  - then we cannot consistently select a *deterministic* fair joint action.

17. Let us take probabilities into account when describing preferences and decisions (cont-d)

- However, we will show that:
  - if we take into account preferences between *probabilistic* options and allow *probabilistic* joint decisions,
  - then a fair solution is possible.

18. Let us take probabilities into account when describing preferences

- First, we need to show how we can describe the corresponding preferences.
- Let us select two alternatives:
  - a very good alternative A_+ which is better than anything that we will actually encounter and
  - a very bad alternative A_- which is worse than anything that we will actually encounter.
- Then, for each real number p from the interval $[0, 1]$, we can consider a probabilistic alternative ("lottery") $L(p)$ in which
  - we get A_+ with probability p, and
  - we get A_- with the remaining probability $1 - p$.

19. Let us take probabilities into account when describing preferences (cont-d)

- For each actual alternative A, we can ask the user to compare:
  - this alternative A
  - with lotteries $L(p)$ corresponding to different probabilities p.
- For small p, the lottery is close to A_- and is, thus, worse than A; we will denote it by $L(p) < A$.
- For probabilities close to 1, the lottery $L(p)$ is close to A_+ and is, thus, better than A: $A < L(p)$.
- Also, if $L(p) < A$ and $p' < p$, then clearly $L(p') < A$.
- Similarly, if $A < L(p)$ and $p < p'$, then $A < L(p')$.

20. Let us take probabilities into account when describing preferences (cont-d)

- One can show that in this case, there exists a threshold value u such that:
  - for all $p < u$, we have $L(p) < A$, while
  - for all $p > u$, we have $A < L(p)$:

$$\inf\{p : L(p) < A\} = \sup\{p : A < L(p)\}.$$

- In this case, for arbitrarily small $\varepsilon > 0$, we have

$$L(u - \varepsilon) < A < L(u + \varepsilon).$$

- In practice, we can only set up probability with some accuracy.
- So this means that, in effect, the alternative A is equivalent to $L(u)$.
- This threshold value u is called the *utility* of the alternative A.
- Utility of A is denoted by $u(A)$.

21. Utility is not uniquely determined

- The numerical value of the utility depends on the selection of the two alternatives A_- and A_+.
- One can show that if we select two different alternatives A'_- and A'_+, then:
  - the new utility value is related to
  - the original utility value by a linear transformation: $u'(A) = a \cdot u(A) + b$ for some $a > 0$ and b.
- One can also show that:
  - the utility of a situation in which we get alternatives A_i with probabilities p_i
  - is equal to $p_1 \cdot u(A_1) + p_2 \cdot u(A_2) + \ldots$

22. How to make a group decision

- Suppose now that n participants need to make a joint decision.
- There is a status quo state A_0 – the state that will occur if we do not make any decision.
- Utility of each participant is defined modulo a linear transformation.
- So, we can always apply a shift $u(A) \mapsto u(A) - u(A_0)$,
- Thus, we get the utility of the status quo state to be equal to 0.
- Therefore, preferences of each participant i can be described by the utility $u_i(A)$ for which $u_i(A_0) = 0$.
- These utilities are defined modulo linear transformations $u_i(A) \mapsto a_i \cdot u_i(A)$ for some a_i.
- It therefore makes sense to require that the group decision making should not change if we thus re-scale each utility value.

23. How to make a group decision (cont-d)

- It turns out that:
  - the only decision making that does not change under this re-scaling
  - means selecting an alternative A with the largest product of the utilities $u_1(A) \cdot \ldots \cdot u_n(A)$.
- This result was first shown by a Nobelist John Nash and is thus known as Nash's bargaining solution.
- If we allow probabilistic combinations of original decisions, then such a solution is, in some reasonable sense, unique.

24. Adding probabilities to Arrow's setting: conclusion

- One can show that Nash's bargaining solution satisfies all fairness requirements.
- So, in this case, we indeed have a solution to the group decision problem.
- This is exactly what, according to Arrow's result, is not possible if we do not take probabilities into account.

25. What about Goedel's theorem?

- Goedel's theorem states that:
  - no matter what finite list of axioms we choose, for some true statements of the type $\forall n\, P(n)$,
  - we will never deduce the truth of this statement from these axioms.
- Let us look at this situation from the commonsense viewpoint.
- The fact that the statement $\forall n\, P(n)$ is true means that for all natural numbers n, we can check that the property $P(n)$ is true.
- We can check it for $n = 0$, we can check it for $n = 1$, etc., and this will be always true.

26. What about Goedel's theorem (cont-d)

- And this is exactly how we reason about the properties of the real world:
  - someone proposes a new physical law,
  - we test it many times,
  - every time, this law is confirmed,
  - so we start believing that this law is true.
- The more experiments we perform, the higher our subjective probability that this law is true.
- If a law is confirmed in n experiments, statistics estimates it as $1-1/n$.
- So, yes, we can never become 100% sure (based on the axioms) that the statement $\forall n\, P(n)$ is true.
- However, are we ever 100% sure?

27. What about Goedel's theorem (cont-d)

- Even for long proofs, there is always a probability that we missed a mistake.
- And sometimes such mistakes are found.
- We usually trust results of computer computations.
- However, sometimes, some of the computer cells goes wrong, and this makes the results wrong.
- Such things also happened.
- So, in this case too:
  - allowing probabilistic outcomes
  - also allows us to make practically definite conclusions.

28. What about Goedel's theorem (cont-d)

- This is contrary to what happens when we are not taking probabilities into account.
- In that case, Goedel's theorem shows that conclusions are, in general, not possible.

29. Acknowledgments

This work was supported in part by:

- National Science Foundation grants 1623190, HRD-1834620, HRD-2034030, and EAR-2225395;
- AT&T Fellowship in Information Technology;
- program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and
- a grant from the Hungarian National Research, Development and Innovation Office (NRDI).