The only award system that prevents cheating is linear

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1. Situations that motivated this research

- Many Gulag memoirs describe the award system that was typically used for the prisoners involved in hard work.
- An example of hard work is felling trees in the Russian North.
- There was a fixed amount of daily food (e.g., 600 grams of bread) to those who fulfilled the norm.
- Those who did not fulfil the norm got much less – e.g., 300 grams.
- Those exceeded the norm by a significant percentage got a larger amount – e.g., 900 grams.
- The norm was based on the ability of people accustomed to hard menial labor.
- Many prisoners who were accustomed to such hard word – e.g., scientists, engineers, actors.

2. Situations that motivated this research (cont-d)

- They were arrested for telling jokes or for having contacts with foreigners.
- Some for them were arrested even for having previous contacts with folks that have been later arrested.
- For them, it was practically impossible to fulfil the norm.
- One of such prisoners, Valerii Frid – later became an award-winning movie maker.
- remembers that he first got into a team led by an inexperienced leader who honestly reported what everyone has achieved.
- As a result of this. everyone got a decreased amount of food, barely enough to stay alive.
- Later on, Frid's team leader learned from his colleagues that it was possible to increase the overall amount of bread allocated to his team.

3. Situations that motivated this research (cont-d)

- For that, instead of reporting the true outcomes, he would allocate the whole output to a few selected folks.
- These selected folks would then be classified as having fulfilled the norm.
- Then the resulting overall amount of bread would be distributed among the whole team.
- For example, suppose that in the team of 20 everyone could maintain only 80% of the norm.
- Then, in the original arrangement, they would get 30×300 grams $=$ 6 kg of bread.
- In the new arrangement, the team's production was packaged as produced by 16 folks each of which thus fulfils the norm.
- Each of these 16 folks would then get 600 grams of bread, to the total of 16×600 grams $= 9.6$ kg.

4. Situations that motivated this research (cont-d)

- The remaining 4 folks would still be allocated 300 grams each, to the total of 4×300 grams $= 1.2$ kg.
- Thus, overall the team would get $9.6 + 1.2 = 10.8$ kg of bread – almost twice larger than in the original arrangement.

5. Comment

- Similar cheating was known to happen in other situations.
- For example, to increase birth rates, special bonuses were given to mothers who have many children.
- So, in very poor areas, several desperate women would claim all their children to be children of one of them.
- Then, they would divide the resulting bonus.

6. A natural question

- Clearly, the Gulag's award system encouraged cheating.
- In the sense that a fictitious redistribution of the overall production between team members could increase the overall award amount.
- So, a natural question is: which award systems prevents such cheating?
- In other words, in which award systems such a fictitious redistribution would not increase the overall award amount?

7. Let us formulate the problem in precise terms

- We consider award systems in which each person's award depends on this person's stated productivity $x \geq 0$.
- Let us denote the amount of award corresponding to productivity x by $f(x)$.
- The purpose of the award system is to encourage workers to increase production.
- So the award for a non-zero productivity should be larger than or equal to the award for zero productivity.
- Thus, we shall have $f(x) \geq f(0)$ for all $x \geq 0$. Hence, we arrive at the following definition.
- *By an* award system *we mean a function from* $[0, \infty)$ *to* $[0, \infty)$ *for which* $f(x) \geq f(0)$ *for all* x*.*

8. Let us formulate the problem in precise terms (cont-d)

- In this context, cheating is possible when after replacing the original productivity values $x_1, \ldots, x_n$ with some fictitious values $y_1, \ldots, y_n$:
  - while overall productivity is the same – i.e., for which $x_1 + \ldots + x_n = y_1 + \ldots + y_n$,
  - we get a larger overall award:

$$f(y_1) + \ldots + f(y_n) > f(x_1) + \ldots + f(x_n).$$

- *We say that an award system* $f(x)$ encourages cheating *if there exists values* $x_1, \ldots, x_n$ *and* $y_1, \ldots, y_n$ *for which*

$$x_1 + \ldots + x_n = y_1 + \ldots + y_n \text{ and } f(y_1) + \ldots + f(y_n) > f(x_1) + \ldots + f(x_n).$$

- *We say that an award system* $f(x)$ prevents cheating *if it does not encourage cheating.*

9. Main result and its proof

- **Proposition.** *An award system $f(x)$ prevents cheating if and only if it is linear: $f(x) = c_0 + c_1 \cdot x$ for some $c_0 \geq 0$ and $c_1 \geq 0$.*
- **Proof.** Let us first prove that linear award systems prevent cheating.
- Indeed, for a linear award system $f(x) = c_0 + c_1 \cdot x$, for all possible tuples $x_1, \ldots, x_n$, we have

  $$f(x_1)+\ldots+f(x_n) = c_0+c_1\cdot x_1+\ldots+c_0+c_1\cdot x_n = n\cdot c_0+c_1\cdot(x_1+\ldots+x_n).$$

- Thus, if we have $x_1 + \ldots + x_n = y_1 + \ldots + y_n$, then we will have

  $$f(y_1) + \ldots + f(y_n) = f(x_1) + \ldots + f(x_n).$$

- So cheating will not be encouraged.
- To complete the proof of the proposition, we need to prove that if an award system $f(x)$ prevents cheating, then it is linear.

10. Proof (cont-d)

- Indeed, for every two non-negative numbers a and b, let us consider the following two arrangements with the same sum:
  - the arrangement $x_1 = a$, $x_2 = b$, and $x_3 = \ldots = x_n = 0$, and
  - the arrangement $y_1 = a + b$ and $y_2 = \ldots = y_n = 0$.
- In both arrangement, we gave $x_1 + \ldots + x_n = y_1 + \ldots + y_n = a + b$.
- Let us now compare the corresponding overall awards

$$f(x_1) + \ldots + f(x_n) \text{ and } f(y_1) + \ldots + f(y_n).$$

- If the 2nd overall award is larger than the 1st one, this would mean that going from $x_1, \ldots, x_n$ to $y_1, \ldots, y_n$ encourages cheating.
- If the 2nd overall award is smaller than the 1st one, this would mean that going from $y_1, \ldots, y_n$ to $x_1, \ldots, x_n$ encourages cheating.
- The second overall award cannot be larger than the first one and cannot be smaller than the first one.

11. Proof (cont-d)

- So, these two overall awards must be equal:

$$f(x_1) + \ldots + f(x_n) = f(y_1) + \ldots + f(y_n).$$

- Substituting the above values of x_i and y_i into this equality, we conclude that

$$f(a) + f(b) + (n-2) \cdot f(0) = f(a+b) + (n-1) \cdot f(0).$$

- Subtracting $n \cdot f(0)$ from both sides, we conclude that

$$f(a) + f(b) - 2f(0) = f(a+b) - f(0).$$

- This is equivalent ro

$$(f(a) - f(0)) + (f(b) - f(0)) = f(a+b) - f(0).$$

- In other words, for the function $f(a) \stackrel{\text{def}}{=} f(a) - f(0)$, we have

$$F(a) + F(b) = F(a+b).$$

12. Proof (cont-d)

- Since we have $f(a) \geq f(0)$ for all a, we thus have $F(a) \geq 0$ for all a.
- It is known that every non-negative solution of the equation $F(a) + F(b) = F(a+b)$ is a linear function: $F(a) = c_1 \cdot a$ for some $c_1 \geq 0$.
- Since $F(a) = f(a) - f(0)$, we thus conclude that $f(a) = f(0) + F(a) = f(0) + c_1 \cdot a$, i.e., that

$$f(a) = c_0 + c_1 \cdot a.$$

- Here we denoted $c_0 \stackrel{\text{def}}{=} f(0)$.
- The proposition is proven.

13. Acknowledgments

This work was supported by:

- the AT&T Fellowship in Information Technology,
- the Institute for Risk and Reliability, Leibniz Universitaet Hannover, Germany,
- the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Focus Program SPP 100+ 2388, Grant Nr. 501624329,
- the European Union under the project ROBOPROX (No. CZ.02.01.01/00/22 008/0004590),
- the Center of Excellence in Econometrics, Faculty of Economics, Chiang Mai University, Thailand,
- the Ho Chi Minh City University of Banking, Vietnam, and
- Thang Long University, Hanoi, Vietnam.