Geometry of Gaudí Arches: Why Parabolic and Catenary Shapes?

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1. Empirical fact

- The famous Catalan architect Antoni Gaudí (1852–1926) had arches in many of his buildings.
- A recent book has shown that practically all his arches have one of the following two shapes.
- Some of them are *parabolic* arches, in which the y-coordinate is a quadratic function of x.
- Gaudí used these arches in Nau Gaudi in Mataró, Collegi de les Tresianes, Palau Gëll, stables at the Finca Güell, and Celler Güell.
- Some are so-called *catenary* arches, described by the formula

$$y = c \cdot \cosh\left(\frac{x}{c}\right) = \frac{c}{2} \cdot (e^{x/c} + e^{-x/c}).$$

- Gaudí used these arches in Pavellons de la Finca Güell, Els Pavellons de la Finca Güell, Cas Battló, Casa Milá, and in Celler Güell.

Figure 1: Palau Güell, Barcelona; photo by Thomas Ledl.

2. A natural question

- How can we explain why Gaudí used only these two types of arches?
- There are some explanations, but the problem largely remains.
- One explanation is that the parabolic arches have limited support ability.
- So, from the mechanical viewpoint, when more support is needed, it is better to use catenary arches.
- This explains why not all Gaudí arches are parabolic.
- However, it does not explain why namely catenary arches are used.
- Indeed, other shapes could also provide additional stability.
- In this talk, we provide a possible explanation of why these two types of arches were selected.

3. Our explanation: main idea

- Our main idea is to take into account that all arch forms are smooth, so the corresponding equations can be described by Taylor series

$$y = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_n \cdot x^n + \ldots$$

- Thus, a good approximation can be obtained if we consider the first few terms of these Taylor series and ignore the higher order terms:

$$y = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_n \cdot x^n.$$

- This idea is ubiquitous:
  - in physics: this is how physicists analyze new phenomena, and
  - in computations: this is how computers usually compute $\sin(x)$, $\exp(x)$, and all other special functions.

4. Let us simplify the formulas by selecting appropriate coordinates

- In order to make our analysis easier, let us first simplify the general formula.
- Let us select, as the starting point $(0, 0)$ of the coordinate system, the top point of the arch.
- Let x-coordinate be the horizontal line.
- Let y coordinate be the vertical line – going upside down, to make sure all the y-values are positive.
- Let us see how this affects the general formula.
- In the selected coordinate system, the curve $y(x)$ goes through the point $(0, 0)$.
- This means that the values $x = y = 0$ should satisfy the above formula.

5. Let us simplify the formulas by selecting appropriate coordinates (cont-d)

- Substituting $x = 0$ and $y = 0$ into this formula, we conclude that $a_0 = 0$.
- The point $(0, 0)$ is the highest point on the arch.
- Taking into account that the coordinate y goes upside down, this means that the value $y = 0$ is the smallest y-value on this curve.
- Thus, according to calculus, the derivative y' of the function y at this point $x = 0$ is equal to 0.
- From the above formula, one can see that this derivative is equal to a_1.
- So, we can conclude that $a_1 = 0$.
- Thus, in the selected coordinates, we have $a_0 = a_1 = 0$, and the above expression takes the following simplified form

$$y = a_2 \cdot x^2 + \ldots + a_n \cdot x^n + \ldots$$

6. Let us simplify the formulas by selecting appropriate coordinates (cont-d)

- In contrast to the case of the first two coefficients a_0 and a_1, there is no reason to believe that $a_2 = 0$.
- The generic value of a_2 is $a_2 \neq 0$, so we will assume that $a_2 \neq 0$.
- Since all y-values are positive, this implies that $a_2 > 0$.

7. Scale-invariance leads to a parabolic arch

- To decide which of the possible shapes to use, let us use some general physics ideas.
- From the physical viewpoint, one of the natural requirements is *scale-invariance*.
- This means that a physical process does not change when we re-scale it.
- Re-scaling means that we design a new system with the same numerical sizes but measured in different units – e.g., in centimeters instead of meters.
- Then, the physical quantities of this system would also take the same numerical values – but in correspondingly changed units.
- For example, in the past, airplanes were tested by placing smaller-size models in a wind tunnel.
- Then, we re-scale back the measured forces and accelerations.

8. Scale-invariance leads to a parabolic arch (cont-d)

- This way, we get a very good picture of how the actual-size airplane will behave in flight.
- In our case, scale-invariance means that:
  - for each re-scaling of the x-coordinate, i.e., for each transformation $x \to X = \lambda \cdot x$ (for some $\lambda > 0$),
  - there exists an appropriate re-scaling $y \to Y = \mu \cdot y$ for which the dependence $Y(X)$ has the same form as $y(x)$.
- In other words, for every $\lambda > 0$, there must exist a value $\mu > 0$ for which

$$y = a_2 \cdot x^2 + a_3 \cdot x^3 + a_4 \cdot x^4 + \ldots \text{ implies that}$$

$$Y = a_2 \cdot X^2 + a_3 \cdot X^3 + a_4 \cdot X^4 + \ldots, \text{ where } X = \lambda \cdot x \text{ and } Y = \mu \cdot y.$$

- Substituting these expressions for X and Y into the above formula, we conclude that

$$\mu \cdot y = a_2 \cdot \lambda^2 \cdot x^2 + a_3 \cdot \lambda^3 \cdot x^3 + a_4 \cdot \lambda^4 \cdot x^4 + \ldots$$

9. Scale-invariance leads to a parabolic arch (cont-d)

- Dividing both sides by μ, we conclude that

$$y = a_2 \cdot \frac{\lambda^2}{\mu} \cdot x^2 + a_3 \cdot \frac{\lambda^3}{\mu} \cdot x^3 + a_4 \cdot \frac{\lambda^4}{\mu} \cdot x^4 + \ldots$$

- So, for all x, we must have the following equality:

$$a_2 \cdot x^2 + a_3 \cdot x^3 + a_4 \cdot x^4 + \ldots = a_2 \cdot \frac{\lambda^2}{\mu} \cdot x^2 + a_3 \cdot \frac{\lambda^3}{\mu} \cdot x^3 + a_4 \cdot \frac{\lambda^4}{\mu} \cdot x^4 + \ldots$$

- It is well known that the coefficients of the Taylor series are uniquely determined by the function; so:
  - if two expressions lead always to the same function,
  - this means that all the corresponding coefficients in the two expansions must coincide.
- By comparing the coefficients at x^2, we conclude that $a_2 = a_2 \cdot \dfrac{\lambda^2}{\mu}$.

10. Scale-invariance leads to a parabolic arch (cont-d)

- Since $a_2 > 0$, we can divide both sides by a_2 and conclude that $\dfrac{\lambda^2}{\mu} = 1$, i.e., that $\mu = \lambda^2$.
- Substituting this expression for μ into the above equality, we conclude that

  $$a_2 \cdot x^2 + a_3 \cdot x^3 + a_4 \cdot x^4 + \ldots = a_2 \cdot x^2 + a_3 \cdot \lambda \cdot x^3 + a_4 \cdot \lambda^2 \cdot x^4 + \ldots$$

- By comparing the coefficients at x^3, we conclude that $a_3 = a_3 \cdot \lambda$.
- Since, in general, $\lambda \neq 1$ – there is no sense to consider re-scaling that does not change anything – we thus conclude that $a_3 = 0$.
- Similarly, we conclude that $a_4 = a_5 = \ldots = 0$, i.e., that $y = a_2 \cdot x^2$.
- So, indeed, the only scale-invariant arch shape is the shape of the parabola.
- This provides some explanation of why parabolic arches are ubiquitous.

11. Comment

- If we did not require that $a_2 \neq 0$, then we would also get the formulas $y = a_4 \cdot x^4$, $y = a_6 \cdot x^6$, etc.
- In general, we get $a_{2n} \cdot x^{2n}$ for some positive integer n.
- Indeed, it is known that the only continuous scale-invariant functions have the form $c \cdot x^k$ for some c and k.
- We are only considering functions that can be expanded in Taylor series.
- So, they must be a linear combinations of x raised to non-negative integer powers.
- This would imply that k should be a non-negative integer.
- The fact that the value y should be non-negative for all x implies that k cannot be odd, it must be even.

12. Let us simplify the formulas by assuming symmetry and by selecting an appropriate unit of distance

- Let us now go back to the general formula.
- Parabolic arches are symmetric with respect to changing the direction of the x-axis: for them, $y(x) = y(-x)$.
- It makes sense to consider symmetric arches in general.
- With respect to general formula, this means that there should be no odd-power terms in the formula, i.e., that we should have

$$y = a_2 \cdot x^2 + a_4 \cdot x^4 + \dots$$

- Also, the above discussion of the possibility of changing the measuring unit brings in the possibility to further simplify this formula.

13. Let us simplify the formulas by assuming symmetry and by selecting an appropriate unit of distance (cont-d)

- Specifically, let us show that we can always select the new measuring unit for distance for which:
  - for the re-scaled values $X = \lambda \cdot x$ and $Y = \lambda \cdot y$,
  - the coefficient a_2 will become equal to 1.
- Indeed, since $Y = \lambda \cdot y$, we have

$$Y = \lambda \cdot y = \lambda \cdot a_2 \cdot x^2 + \lambda \cdot a_4 \cdot x^4 + \ldots$$

- Substituting $x = X/\lambda$ into this formula, we conclude that

$$Y = \lambda \cdot a_2 \cdot \frac{X^2}{\lambda^2} + \lambda \cdot a_4 \cdot \frac{X^4}{\lambda^4} + \ldots = \frac{a_2}{\lambda} \cdot X^2 + \frac{a_4}{\lambda^3} \cdot X^4 + \cdots$$

- Thus, for $\lambda = a_2$, we indeed get an expression with $a_2 = 1$.

14. Let us simplify the formulas by assuming symmetry and by selecting an appropriate unit of distance (cont-d)

- So, without losing generality, we can assume that we are using the corresponding new units.
- In these units, $a_2 = 1$ and the desired dependence has the form

$$y = x^2 + a_4 \cdot x^4 + \ldots$$

15. What value a_4 should we choose?

- The main idea behind using the Taylor expansion is that we should have a convergent sum.
- If we take $a_4 = 1$, this starts looking like a geometric progression $x^2 + x^4 + x^6 + \ldots$ that does not converge when $|x| \geq 1$.
- Similarly, if we take $a_4 = -1$, this starts looking like a geometric progression $x^2 - x^4 + x^6 - \ldots$ that also stops converging when $|x| = 1$.
- So, to increase the chances of convergence, it is reasonable to take the values a_4 which are closer to 0 that these two extreme values $a_4 = \pm 1$.
- So, we require that $|a_4| < 1$.
- Values close to 1 converge slower, so it makes sense to select a value $|a_4|$ which is:
  - closer to the value $a_4 = 0$ (that corresponds to the parabolic arch)
  - than to the threshold value $|a_4| = 1$ at which the series stops converging.

16. What value a_4 should we choose (cont-d)

- In other words, the distance $||a_4| - 0| = |a_4|$ between $|a_4|$ and 0 should be smaller than distance $||a_4| - 1|$ between $|a_4|$ and 1.
- In general, what is the reasonable choice of two positive quantities a and b:
  - if all we know is that $a < b$,
  - i.e., that the ratio a/b is somewhere between 0 and 1?
- Based on this information, there is no reason to expect that some ratios a/b are more probable.
- It is therefore reasonable to assume:
  - that all the values from the interval $[0, 1]$ are equally probable,
  - i.e., in precise terms, that we have a uniform distribution on the interval $[0, 1]$.

17. What value a_4 should we choose (cont-d)

- This argument is a particular case of the general Laplace Indeterminacy Principle, according to which:
  - if we have no reason to believe that one of the situations is more probable,
  - it make sense to assume that they are equally probable.
- Our knowledge does not change if we swap two alternatives, so the probability distribution based on this knowledge should also be similarly invariant.
- In general, if we only know the probability distribution, then a reasonable number representing this distribution is its mean.
- For the uniform distribution, the mean value is $1/2$.
- Sometimes, the median is a better representation, but for the uniform distribution, the median is also $1/2$.

18. What value a_4 should we choose (cont-d)

- So, it makes sense to assume that value $|a_4|$ is exactly $1/2$ of the value $1 - |a_4|$, i.e., that $|a_4| = 0.5 \cdot |1 - |a_4||$.
- Let us see what it implies depending on the sign of the difference $1 - |a_4|$.
- When $1 - |a_4| > 0$, this equality implies that $|a_4| = 0.5 \cdot (1 - |a_4|)$, i.e., that $|a_4| = 1/3$.
- When $1 - |a_4| < 0$, this implies that $|a_4| = |a_4| - 1$, i.e., subtracting $|a_4|$ from both sides, that $0 = -1$, which is not possoble.
- Thus, we have $|a_4| = 1/3$, i.e., $a_4 = \pm 1/3$.
- Let us recall that to make the arch more stable, we need to increase a_4.

19. What value a_4 should we choose (cont-d)

- So, we can ignore the value $a_4 = -1/3$ that leads to an even less stable arch, and only consider the value $a_4 = 1/3$.
- For this value, the general formula takes the form

$$y = x^2 + \frac{1}{3} \cdot x^4 + \ldots$$

20. Interestingly, this formula corresponds exactly to catenary arches

- To show this, let us expand the expression for the catenary arch in Taylor series.
- To do this, let us recall the Taylor series for e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$$

- Thus,

$$e^{x/c} = 1 + \frac{x}{c} + \frac{x^2}{2c^2} + \frac{x^3}{6c^3} + \frac{x^4}{24c^4} + \ldots, \text{ and}$$

$$e^{-x/c} = 1 - \frac{x}{c} + \frac{x^2}{2c^2} - \frac{x^3}{6c^3} + \frac{x^4}{24c^4} + \ldots$$

- By adding these two expressions, we conclude:
  - that the odd-power terms in the sum $e^{x/c} + e^{-x/c}$ cancel each other, and
  - that the even-power terms double.

21. Interestingly, this formula corresponds exactly to catenary arches (cont-d)

– So:

$$y = \frac{c}{2}\cdot(e^{x/c}+e^{-x/c}) = 2\cdot\frac{c}{2}\cdot\left(1+\frac{x^2}{2c^2}+\frac{x^4}{24c^4}+\ldots\right) = c+\frac{x^2}{2c}+\frac{x^4}{24c^3}+\ldots$$

- Let us transform this formula to our selected coordinates.
- First, we selected the starting point of the coordinate system to be the point $(0,0)$, while the above formula for $x=0$ leads to $y=c$.
- To transform this formula to our coordinates, we thus need to change the starting point for y, i.e., to subtract c from all the y-values.
- As a result, we get $y = \dfrac{x^2}{2c}+\dfrac{x^4}{24c^3}+\ldots$
- Second, we have selected the measuring unit for which the coefficient at x^2 is equal to 1.

22. Interestingly, this formula corresponds exactly to catenary arches (cont-d)

- To transform into these coordinates, we need to select the scaling coefficient c in the above formula for which $1/(2c) = 1$.
- So, $2c = 1$ and $c = 1/2$.
- For this c, we have $24c^3 = 24 \cdot (1/2)^3 = 24/8 = 3$, so the above formula takes the form: $y = x^2 + \dfrac{x^4}{3} + \ldots$
- This is exactly the formula that we came up with based on several natural assumptions.
- So, these assumptions justify the use of catenary arches.

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