For Quantum and Reversible Computing, Intervals Are More Appropriate Than General Sets, And Fuzzy Numbers Than General Fuzzy Sets

Oscar Galindo and Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso, El Paso, Texas 79968, USA
ogalindomo@miners.utep.edu, vladik@utep.edu

1. Need for Quantum Computing

- Our current computers are very fast in comparison with what was available a few years ago.
- However, there are still computational tasks that necessitate even faster computers.
- To speed up computers, we need to squeeze in more cells and into the same volume.
- For that, we need to make cells as small as possible.
- Already, the existing cells contain a small number of molecules.
- If we decrease them further, they will contain a few molecules.
- Thus, we will need to take into account quantum effects.

2. Quantum Computing: Additional Advantages

- There are innovative algorithms specifically designed for quantum computing.
- We can decrease the time needed to find an element in an unsorted array of size n from n to $\sqrt{n}$ steps.
- We can reduce the time needed to factor large integers of n digits from exponential to polynomial in n.
- This task is needed to decode currently encoded messages.

3. Need for Reversible Computing

- One challenge in designing quantum computers is that on the quantum level, all equations are time-reversible.
- In the traditional algorithms, even the simplest "and"-operation $a, b \to a \,\&\, b$ is not reversible:
  - if we know its result $a \,\&\, b = 0 =$ "false",
  - we cannot uniquely reconstruct the input (a, b).
- Reversibility is also important because, according to statistical physics:
  - any irreversible process means increasing entropy,
  - and this leads to heat emission.
- Overheating is one of the reasons why we cannot pack too many processing units into the same volume.
- So, to pack more, it is desirable to reduce this heat emission – e.g., by using only reversible computations.

4. Need to Take Uncertainty into Account

- We use computers mostly to process data.
- When processing data, we need to take into account that data comes from measurements.
- Measurements are never absolutely accurate.
- The measurement result $\widetilde{x}$ is, in general, different from the actual value x of the corresponding quantity.
- It is therefore necessary to take this uncertainty into account when processing data.

5. Need for Interval Uncertainty

- In many real life situations:
  - the only information that we have about the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$ is
  - the upper bound Δ on its absolute value:
  $$|\Delta x| \le \Delta.$$
- Once we have a measurement result $\widetilde{x}$, then:
  - the only information that we can conclude about the actual value x is that
  - this value is somewhere in the interval $[\widetilde{x}-\Delta, \widetilde{x}+\Delta]$.
- Such interval uncertainty indeed appears in many practical applications.

6. Data Processing under Interval Uncertainty

- In a data processing algorithm:
  - we take several inputs $x_1, \ldots, x_n$, and
  - we apply an appropriate algorithm to generate the result y depending on these inputs.
- Let us denote this dependence by $f(x_1, \ldots, x_n)$.
- For each input i, we only know the interval $X_i = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$ of possible values of x_i.
- Then, the only information that we can have about y is that y belongs to the set
$$Y = f(X_1, \ldots, X_n) \stackrel{\text{def}}{=}$$
$$\{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}.$$
- When the sets X_i are intervals and the function $f(x_1, \ldots, x_n)$ is continuous, the resulting set Y is also an interval.
- In most practical situations, the measurement errors are relatively small.
- So, we can expand the function $f(x_1, \ldots, x_n)$ in Taylor series and retain only linear terms.
- Then, we get
$$f(x_1, \ldots, x_n) = f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n) \approx$$
$$\widetilde{y} - \sum_{i=1}^{n} c_i \cdot \Delta x_i, \quad \widetilde{y} \stackrel{\text{def}}{=} f(\widetilde{x}_1, \ldots, \widetilde{x}_n), \quad c_i \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_i}_{|x_i = \widetilde{x}_i}.$$
- In other words, $f(x_1, \ldots, x_n)$ becomes a linear function:
$$f(x_1, \ldots, x_n) = c_0 + \sum_{i=1}^{n} c_i \cdot x_n, \quad c_0 \stackrel{\text{def}}{=} \widetilde{y} - \sum_{i=1}^{n} c_i \cdot \widetilde{x}_i.$$
- In other words, data processing can be, in effect, reduced to multiplication by a constant c_i and addition.

7. When Is This Data Processing Reversible?

- Multiplication by a constant is always reversible.
- Indeed, if we know the interval $Y = c \cdot X$, then, we can reconstruct X as $X = c^{-1} \cdot Y$.
- Addition $y = x_1 + x_2$ is also reversible.
- Indeed, if we know that $x_1 \in [\underline{x}_1, \overline{x}_1]$ and $x_2 \in [\underline{x}_2, \overline{x}_2]$, then $Y = [\underline{y}, \overline{y}]$ has the form
$$Y = [\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2].$$
- If we know $Y = [\underline{y}, \overline{y}]$ and $X_1 = [\underline{x}_1, \overline{x}_1]$, then we can reconstruct $X_2 = [\underline{x}_2, \overline{x}_2]$ as
$$\underline{x}_2 = \underline{y} - \underline{x}_1 \text{ and } \overline{x}_2 = \overline{y} - \overline{x}_1.$$

8. From Interval Uncertainty to a More General Set Uncertainty

- In some cases:
  - in addition to knowing that values of x are within a certain interval $[\underline{x}, \overline{x}]$,
  - we also know that some values from this interval are not possible.
- In this case, the set X of possible values of x is different from an interval.
- No matter how crude the measurements are, there is always an upper bound Δ on the measurement error.
- Thus, all possible values of x are in the interval
$$[\widetilde{x} - \Delta, \widetilde{x} + \Delta].$$
- Thus, the set X is bounded.
- In general, we can safely assume that the set X is closed.
- Indeed, suppose that x_0 is a limit point of the set.
- Then, for every $\varepsilon > 0$, there are elements $x \in X$ is any ε-neighborhood $(x_0 - \varepsilon, x_0 + \varepsilon)$ of this value x_0.
- This means that:
  - no matter how accurately we measure the corresponding value,
  - we will not be able to distinguish between the limit value x_0 and a sufficient close value $x \in X$.
- It is therefore reasonable to simply assume that x_0 is possible.
- Thus, we conclude that the set of possible values of x contains all its limit points, i.e., is closed.

9. Data Processing under Set Uncertainty

- Assume that we know the set X_1 of possible values of x_1, and we know the set X_2 of possible values of x_2.
- Then the set $Y \stackrel{\text{def}}{=} X_1 + X_2$ of possible values of the sum $y = x_1 + x_2$ is equal to
$$Y = \{x_1 + x_2 : x_1 \in X_1 \text{ and } x_2 \in X_2\}.$$
- If we add any non-interval bounded closed set S to the class of all intervals, additions stops being reversible.
- For $\underline{S} \stackrel{\text{def}}{=} \inf\{x : x \in S\}$ and $\overline{S} \stackrel{\text{def}}{=} \sup\{x : x \in S\}$, we have
$$[\underline{S}, \overline{S}] + [\underline{S}, \overline{S}] = [\underline{S}, \overline{S}] + S (= [2\underline{S}, 2\overline{S}]).$$
- However, $[\underline{S}, \overline{S}] \ne S$.

10. Case of Fuzzy Uncertainty

- In many real-life situations:
  - in addition to the guaranteed upper bound Δ on the absolute value of the measurement error,
  - with some degree of certainty β, measurement errors can be bounded by a smaller bound $\Delta(\beta) < \Delta$.
- As a result:
  - in addition to the interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$ that is guaranteed to contain x with 100% confidence,
  - we have several narrower intervals $[\widetilde{x} - \Delta(\beta), \widetilde{x} + \Delta(\beta)]$ that contain x with confidence β.
- In other words, we have a nested family of intervals corresponding to different values β.
- The larger the β (i.e., the higher the desired confidence), the wider the interval.

Case of Fuzzy Uncertainty (cont-d)

- Such a family of nested interval is, in effect, an equivalent way of representing a fuzzy number.
- If instead of intervals, we have more general sets $S(\beta)$, then we have a *fuzzy set.*
- The sets $S(\beta)$ are known as α*-cuts* of the fuzzy set, where $\alpha \stackrel{\text{def}}{=} 1 - \beta$.
- For such fuzzy sets, we can define operations layer-by-layer:
  - for each β (i.e., equivalently, for each α),
  - we process all the sets (or intervals) corresponding to this value β.
- Fuzzy numbers correspond to intervals, and general fuzzy sets to general sets.
- So, we conclude that addition is only reversible for fuzzy numbers.
- If we add any fuzzy set which is not a fuzzy number to fuzzy numbers, addition stops being reversible.

11. Intervals are Ubiquitous

- We showed that intervals (and fuzzy numbers) are preferable: they lead to reversible data processing.
- Interestingly, intervals (and fuzzy numbers) are indeed ubiquitous.
- They occur much much more frequently in practice as descriptions of uncertainty than any other sets.
- Why is that?

12. A Possible Explanation: Main Idea

- Let us recall why normal (Gaussian) distributions are ubiquitous.
- The usual explanation is that usually, there are many different independent sources of measurement error.
- As a result, the measurement error is a sum of a large number of small independent random variables.
- In the limit, when the number of terms increases, the distribution of the sum tends to normal.
- This is known as the Central Limit Theorem.
- This means that when the number of components is large, the corresponding distribution is close to normal.
- Thus, from the practical viewpoint, we can safely consider the distribution to be normal.
- In non-probabilistic case, the situation is similar.
- The measurement error is the sum of a large number n of small independent error components:
$$\Delta x = \Delta x^{(1)} + \Delta x^{(2)} + \ldots + \Delta x^{(n)}.$$
- Let us assume that for each of the components $\Delta x^{(k)}$, we know the set $X^{(k)}$ of possible values.
- Then the set S of possible values of their sum is equal to the sum of these sets:
$$X = X^{(1)} + \ldots + X^{(n)} =$$
$$\{\Delta x^{(1)} + \Delta x^{(2)} + \ldots + \Delta x^{(n)} :$$
$$\Delta x^{(1)} \in X^{(1)}, \ldots, \Delta x^{(n)} \in X^{(n)}\}.$$
- It can be shown that, when n increases, the resulting set X also tends to an interval.

13. Need for a More Detailed Explanation

- The limit closeness is good.
- However, in practice, it is desirable to know exactly how close is the resulting set X to an interval.
- For every positive real number $\varepsilon > 0$, two points a and b are ε-close is $|a - b| \le \varepsilon$.
- It is therefore reasonable to say that the sets A and B are ε*-close* if:
  - every point $a \in A$ is ε-close to some point $b \in B$, and
  - every point $b \in B$ is ε-close to some point $a \in A$.
- The smallest value ε with this property is known as the *Hausdorff distance* $d_H(A, B)$ between the two sets.

14. How to Measure Smallness of a Set

- The size of a set A can be naturally measured by its *diameter* $\text{diam}(A)$.
- The diameter is the largest possible distance $d(a, a')$ between the two points a, a' from this set.
- For bounded closed subsets A of a real line, the diameter is equal to $\text{diam}(A) = \sup A - \inf A$.

15. Our Main Result

- *If* $\text{diam}(A_i) \le \varepsilon$ *for all* $i = 1, \ldots, n$, *then for* $A = A_1 + \ldots + A_n$ *and for some interval* I:
$$d_H(A, I) \le \varepsilon/2.$$
- This bound cannot be improved, as shown by the following auxiliary result.
- *For every* n, *there exist closed bounded sets* $A_1, \ldots, A_n$ *for which* $\text{diam}(A_i) \le \varepsilon$ *for all* i, *and for which*
$$d_H(A, I) \ge \varepsilon/2 \text{ for all } I.$$

16. Acknowledgments

This work was partially supported by the US National Science Foundation via grant HRD-1242122 (Cyber-ShARE).