Dealing with Uncertainties in Data Processing: from Probabilistic and Interval Uncertainty to Combination of Different Approaches

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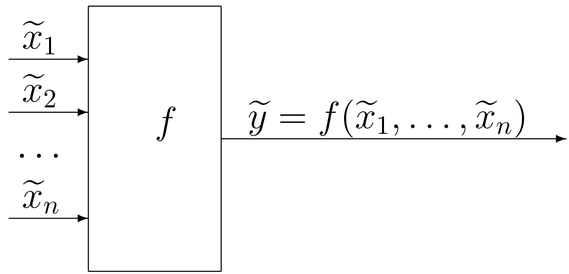
1. General Problem of Data Processing under Uncertainty

- *Indirect measurements:* way to measure y that are difficult (or even impossible) to measure directly.
- *Idea:* $y = f(x_1, \ldots, x_n)$

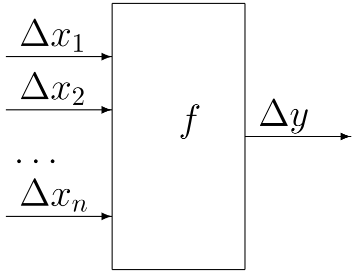


- *Problem:* measurements are never 100% accurate: $\widetilde{x}_i \neq x_i$ ($\Delta x_i \neq 0$) hence

$$\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) \neq y = f(x_1, \ldots, x_n).$$

What are bounds on $\Delta y \stackrel{\text{def}}{=} \widetilde{y} - y$?

2. Probabilistic and Interval Uncertainty

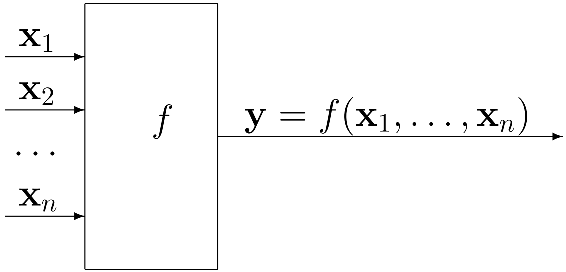


- *Traditional approach:* we know probability distribution for Δx_i (usually Gaussian).
- *Where it comes from:* calibration using standard MI.
- *Problem:* calibration is not possible in:
  - fundamental science
  - manufacturing
- *Solution:* we know upper bounds Δ_i on $|\Delta x_i|$ hence

$$x_i \in [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i].$$

Home Page
Title Page
Page 3 of 80
Go Back
Full Screen
Close
Quit

3. Interval Computations: A Problem



- *Given:* an algorithm $y = f(x_1, \ldots, x_n)$ and n intervals $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$.
- *Compute:* the corresponding range of y:

  $$[\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) \,|\, x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

- *Fact:* NP-hard even for quadratic f.
- *Challenge:* when are feasible algorithms possible?
- *Challenge:* when computing $\mathbf{y} = [\underline{y}, \overline{y}]$ is not feasible, find a good approximation $\mathbf{Y} \supseteq \mathbf{y}$.

Home Page
Title Page
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit

4. Example 1: Formulation of the Problem

- *Given:*
  - $y = f(x_1, x_2) = x_1^2 + x_2^2$,
  - $\widetilde{x}_1 = 1$, $\widetilde{x}_2 = -1$,
  - $\Delta_1 = \Delta_2 = 0.1$.
- In this case, x_1 can be anywhere in

$$[\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1] = [1 - 0.1, 1 + 0.1] = [0.9, 1.1].$$

- Similarly, x_2 can be anywhere in

$$[\widetilde{x}_2 - \Delta_2, \widetilde{x}_2 + \Delta_2] = [-1 - 0.1, -1 + 0.1] = [-1.1, -0.9].$$

- Here, $\widetilde{y} = (\widetilde{x}_1)^2 + (\widetilde{x}_2)^2 = 1^2 + (-1)^2 = 2$.
- *Find:* the range $[\underline{y}, \overline{y}]$ of possible values of y.

5. Example 1: Actual Range

- *Reminder:* $y = x_1^2 + x_2^2 = x_1^2 + |x_2|^2$,

$$x_1 \in [0.9, 1.1], \quad x_2 \in [-1.1, -0.9].$$

- *Actual range:* the function $y = x_1^2 + x_2^2$ is increasing in x_1 and $|x_2|$.
- So, the smallest value of y is

$$\underline{y} = 0.9^2 + (-0.9)^2 = 0.81 + 0.81 = 1.62.$$

- The largest value of y is

$$\overline{y} = 1.1^2 + (-1.1)^2 = 1.21 + 1.21 = 2.42.$$

- Thus, the desired range is $[\underline{y}, \overline{y}] = [1.62, 2.42]$.

6. Exercise 1: Formulation of the Problem

- *Given:*
  - $y = f(x_1, x_2) = x_1^2 - x_1 \cdot x_2 + x_2^2$,
  - $\widetilde{x}_1 = 2$, $\widetilde{x}_2 = -2$,
  - $\Delta_1 = \Delta_2 = 0.2$.
- In this case, x_1 can be anywhere in

$$[\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1] = [2 - 0.2, 2 + 0.2] = [1.8, 2.2].$$

- Similarly, x_2 can be anywhere in

$$[\widetilde{x}_2 - \Delta_2, \widetilde{x}_2 + \Delta_2] = [-2 - 0.2, -2 + 0.2] = [-2.2, -1.8].$$

- $\widetilde{y} = (\widetilde{x}_1)^2 - \widetilde{x}_1 \cdot \widetilde{x}_2 + (\widetilde{x}_2)^2 = 2^2 - 2 \cdot (-2) + (-2)^2 = 12.$
- *Find:* the range $[\underline{y}, \overline{y}]$ of possible values of y.

7. Alternative Approach: Maximum Entropy

- *Situation:* in many practical applications, it is very difficult to come up with the probabilities.
- *Traditional engineering approach:* use probabilistic techniques.
- *Problem:* many different probability distributions are consistent with the same observations.
- *Solution:* select one of these distributions – e.g., the one with the largest entropy.
- *Example – single variable:* if all we know is that $x \in [\underline{x}, \overline{x}]$, then MaxEnt leads to a uniform distribution.
- *Example – multiple variables:* different variables are independently distributed.

8. Limitations of Maximum Entropy Approach

- *Example:* simplest algorithm $y = x_1 + \ldots + x_n$.
- *Measurement errors:* $\Delta x_i \in [-\Delta, \Delta]$.
- *Analysis:* $\Delta y = \Delta x_1 + \ldots + \Delta x_n$.
- *Worst case situation:* $\Delta y = n \cdot \Delta$.
- *Maximum Entropy approach:* due to Central Limit Theorem, Δy is $\approx$ normal, with $\sigma = \Delta \cdot \dfrac{\sqrt{n}}{\sqrt{3}}$.
- *Why this may be inadequate:* we get $\Delta \sim \sqrt{n}$, but due to correlation, it is possible that $\Delta = n \cdot \Delta \sim n \gg \sqrt{n}$.
- *Conclusion:* using a single distribution can be very misleading, especially if we want guaranteed results.
- Examples: high-risk application areas such as space exploration or nuclear engineering.

9. Linearization is Usually Possible

- In many practical situations, the errors Δx_i are small, so we can ignore quadratic terms:

$$\Delta y = \widetilde{y} - y = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) - f(x_1, \ldots, x_n) =$$
$$f(\widetilde{x}_1, \ldots, \widetilde{x}_n) - f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n) \approx$$
$$c_1 \cdot \Delta x_1 + \ldots + c_n \cdot \Delta x_n,$$

  where $c_i \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial x_i}(\widetilde{x}_1, \ldots, \widetilde{x}_n)$.

- For a linear function, the largest Δy is obtained when each term $c_i \cdot \Delta x_i$ is the largest:

$$\Delta = |c_1| \cdot \Delta_1 + \ldots + |c_n| \cdot \Delta_n.$$

- Due to the linearization assumption, we can estimate each partial derivative c_i as

$$c_i \approx \frac{f(\widetilde{x}_1, \ldots, \widetilde{x}_{i-1}, \widetilde{x}_i + h_i, \widetilde{x}_{i+1}, \ldots, \widetilde{x}_n) - \widetilde{y}}{h_i}.$$

10. Example 1: Reminder

- *Given:*
  - $y = f(x_1, x_2) = x_1^2 + x_2^2$,
  - $\widetilde{x}_1 = 1$, $\widetilde{x}_2 = -1$,
  - $\Delta_1 = \Delta_2 = 0.1$.
- In this case, x_1 can be anywhere in

$$[\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1] = [1 - 0.1, 1 + 0.1] = [0.9, 1.1].$$

- Similarly, x_2 can be anywhere in

$$[\widetilde{x}_2 - \Delta_2, \widetilde{x}_2 + \Delta_2] = [-1 - 0.1, -1 + 0.1] = [-1.1, -0.9].$$

- Here, $\widetilde{y} = (\widetilde{x}_1)^2 + (\widetilde{x}_2)^2 = 1^2 + (-1)^2 = 2$.
- *Find:* the range $[\underline{y}, \overline{y}]$ of possible values of y.

11. Example 1: Solution by Linearization

- *Reminder:* $y = x_1^2 + x_2^2$, $\widetilde{y} = 2$.
- Here:

$$c_1 = \frac{\partial f}{\partial x_i}(\widetilde{x}_1, \ldots, \widetilde{x}_n) = 2\widetilde{x}_1 = 2 \cdot 1 = 2;$$

$$c_2 = \frac{\partial f}{\partial x_i}(\widetilde{x}_1, \ldots, \widetilde{x}_n) = 2\widetilde{x}_2 = 2 \cdot (-1) = -2.$$

- Thus,

$$\Delta = |c_1| \cdot \Delta_1 + |c_2| \cdot \Delta_2 = 2 \cdot 0.1 + |-2| \cdot 0.1 =$$

$$2 \cdot 0.1 + 2 \cdot 0.1 = 0.4.$$

- *Estimated range:*

$$[\widetilde{y} - \Delta, \widetilde{y} + \Delta] = [2 - 0.4, 2 + 0.4] = [1.6, 2.4].$$

- *Comment.* This is close to the actual range $[1.62, 2.42]$.

12. Exercise 2

- *Given:*
  - $y = f(x_1, x_2) = x_1^2 - x_1 \cdot x_2 + x_2^2$,
  - $\widetilde{x}_1 = 2$, $\widetilde{x}_2 = -2$,
  - $\Delta_1 = \Delta_2 = 0.2$.
- In this case, x_1 can be anywhere in

$$[\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1] = [2 - 0.2, 2 + 0.2] = [1.8, 2.2].$$

- Similarly, x_2 can be anywhere in

$$[\widetilde{x}_2 - \Delta_2, \widetilde{x}_2 + \Delta_2] = [-2 - 0.2, -2 + 0.2] = [-2.2, -1.8].$$

- $\widetilde{y} = (\widetilde{x}_1)^2 - \widetilde{x}_1 \cdot \widetilde{x}_2 + (\widetilde{x}_2)^2 = 2^2 - 2 \cdot (-2) + (-2)^2 = 12.$
- *Task:* use linearization to find the range $[\underline{y}, \overline{y}]$ of possible values of y.

13. Linearization: Algorithm

To compute the range $\mathbf{y}$ of y, we do the following.

- First, we apply the algorithm f to the original estimates $\widetilde{x}_1, \ldots, \widetilde{x}_n$, resulting in the value $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$.
- Second, for all i from 1 to n,
  - we compute $f(\widetilde{x}_1, \ldots, \widetilde{x}_{i-1}, \widetilde{x}_i + h_i, \widetilde{x}_{i+1}, \ldots, \widetilde{x}_n)$ for some small h_i and then
  - we compute

$$c_i = \frac{f(\widetilde{x}_1, \ldots, \widetilde{x}_{i-1}, \widetilde{x}_i + h_i, \widetilde{x}_{i+1}, \ldots, \widetilde{x}_n) - \widetilde{y}}{h_i}.$$

- Finally, we compute $\Delta = |c_1| \cdot \Delta_1 + \ldots + |c_n| \cdot \Delta_n$ and the desired range $\mathbf{y} = [\widetilde{y} - \Delta, \widetilde{y} + \Delta]$.
- *Problem:* we need $n+1$ calls to f, and this is often too long.

14. Example 1: Reminder

- *Given:*
  - $y = f(x_1, x_2) = x_1^2 + x_2^2$,
  - $\widetilde{x}_1 = 1$, $\widetilde{x}_2 = -1$,
  - $\Delta_1 = \Delta_2 = 0.1$.
- In this case, x_1 can be anywhere in

$$[\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1] = [1 - 0.1, 1 + 0.1] = [0.9, 1.1].$$

- Similarly, x_2 can be anywhere in

$$[\widetilde{x}_2 - \Delta_2, \widetilde{x}_2 + \Delta_2] = [-1 - 0.1, -1 + 0.1] = [-1.1, -0.9].$$

- Here, $\widetilde{y} = (\widetilde{x}_1)^2 + (\widetilde{x}_2)^2 = 1^2 + (-1)^2 = 2$.
- *Find:* the range $[\underline{y}, \overline{y}]$ of possible values of y.

15. Example 1: Solution 2 by Linearization

- *Reminder:* $f(x_1, x_2) = x_1^2 + x_2^2$, $\widetilde{y} = 2$.
- Let us take $h_1 = h_2 = 0.1$; then:

$$c_1 = \frac{f(\widetilde{x}_1 + 0.1, \widetilde{x}_2) - \widetilde{y}}{0.1} = \frac{(\widetilde{x}_1 + 0.1)^2 + (\widetilde{x}_2)^2 - 2}{0.1} =$$

$$\frac{1.1^2 + 1 - 2}{0.1} = \frac{1.21 + 1 - 2}{0.1} = \frac{0.21}{0.1} = 2.1;$$

$$c_2 = \frac{f(\widetilde{x}_1, \widetilde{x}_2 + 0.1) - \widetilde{y}}{0.1} = \frac{(\widetilde{x}_1)^2 + (\widetilde{x}_2 + 0.1)^2 - 2}{0.1} =$$

$$\frac{1^2 + (-0.9)^2 - 2}{0.1} = \frac{1 + 0.81 - 2}{0.1} = \frac{-0.19}{0.1} = 1.9.$$

$$\Delta = |c_1| \cdot \Delta_1 + |c_2| \cdot \Delta_2 = 2.1 \cdot 0.1 + |-1.9| \cdot 0.1 =$$

$$2.1 \cdot 0.1 + 1.9 \cdot 0.1 = 0.4.$$

- *Estimated range:*

$$[\widetilde{y} - \Delta, \widetilde{y} + \Delta] = [2 - 0.4, 2 + 0.4] = [1.6, 2.4].$$

- *Comment.* This is close to the actual range $[1.62, 2.42]$.

16. Exercise 3: Reminder

- *Given:*
  - $y = f(x_1, x_2) = x_1^2 - x_1 \cdot x_2 + x_2^2$,
  - $\widetilde{x}_1 = 2$, $\widetilde{x}_2 = -2$,
  - $\Delta_1 = \Delta_2 = 0.2$.
- In this case, x_1 can be anywhere in

$$[\widetilde{x}_1 - \Delta_1, \widetilde{x}_1 + \Delta_1] = [2 - 0.2, 2 + 0.2] = [1.8, 2.2].$$

- Similarly, x_2 can be anywhere in

$$[\widetilde{x}_2 - \Delta_2, \widetilde{x}_2 + \Delta_2] = [-2 - 0.2, -2 + 0.2] = [-2.2, -1.8].$$

- $\widetilde{y} = (\widetilde{x}_1)^2 - \widetilde{x}_1 \cdot \widetilde{x}_2 + (\widetilde{x}_2)^2 = 2^2 - 2 \cdot (-2) + (-2)^2 = 12.$
- *Task:* use linearization to find the range $[\underline{y}, \overline{y}]$ of possible values of y.

17. Cauchy Deviate Method: Idea

- For large n, we can further reduce the number of calls to f if we Cauchy distributions, w/pdf

$$\rho(z) = \frac{\Delta}{\pi \cdot (z^2 + \Delta^2)}.$$

- Known property of Cauchy transforms:
  - if $z_1, \ldots, z_n$ are independent Cauchy random variables w/parameters $\Delta_1, \ldots, \Delta_n$,
  - then $z = c_1 \cdot z_1 + \ldots + c_n \cdot z_n$ is also Cauchy distributed, w/parameter

$$\Delta = |c_1| \cdot \Delta_1 + \ldots + |c_n| \cdot \Delta_n.$$

- This is exactly what we need to estimate interval uncertainty!

18. Cauchy Deviate Method: Towards Implementation

- To implement the Cauchy idea, we must answer the following questions:
  - how to simulate the Cauchy distribution; and
  - how to estimate the parameter Δ of this distribution from a finite sample.
- Simulation can be based on the functional transformation of uniformly distributed sample values:

$$\delta_i = \Delta_i \cdot \tan(\pi \cdot (r_i - 0.5)), \text{ where } r_i \sim U([0,1]).$$

- To estimate Δ, we can apply the Maximum Likelihood Method $\rho(\delta^{(1)}) \cdot \rho(\delta^{(2)}) \cdot \ldots \cdot \rho(\delta^{(N)}) \to \max$, i.e., solve

$$\frac{1}{1+\left(\dfrac{\delta^{(1)}}{\Delta}\right)^2} + \ldots + \frac{1}{1+\left(\dfrac{\delta^{(N)}}{\Delta}\right)^2} = \frac{N}{2}.$$

19. Cauchy Deviates Method: Algorithm

- Apply f to $\widetilde{x}_i$; we get $\widetilde{y} := f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$.
- For $k = 1, 2, \ldots, N$, repeat the following:
  - use the standard RNG to draw $r_i^{(k)} \sim U([0,1])$, $i = 1, 2, \ldots, n$;
  - compute Cauchy distributed values $c_i^{(k)} := \tan(\pi \cdot (r_i^{(k)} - 0.5))$;
  - compute $K := \max_i |c_i^{(k)}|$ and normalized errors $\delta_i^{(k)} := \Delta_i \cdot c_i^{(k)} / K$;
  - compute the simulated "actual values" $x_i^{(k)} := \widetilde{x}_i - \delta_i^{(k)}$;
  - compute simulated errors of indirect measurement: $\delta^{(k)} := K \cdot \left(\widetilde{y} - f\left(x_1^{(k)}, \ldots, x_n^{(k)}\right)\right)$;
- Compute Δ by applying the bisection method to solve the Maximum Likelihood equation.

20. Important Comment

- To avoid confusion, we should emphasize that:
  - in contrast to the Monte-Carlo solution for the probabilistic case,
  - the use of Cauchy distribution in the interval case is a computational trick,
  - it is *not* a truthful simulation of the actual measurement error Δx_i.
- Indeed:
  - we know that the actual value of Δx_i is always inside the interval $[-\Delta_i, \Delta_i]$, but
  - a Cauchy distributed random attains values outside this interval as well.

21. Approximate Methods – Such As Linearizaion – Are Sometimes Not Sufficient

- In many application areas, it is sufficient to have an *approximate* estimate of y.
- Sometimes, we need to *guarantee* that y does not exceed a certain threshold y_0. Examples:
  - in *nuclear engineering*, the temperatures and the neutron flows should not exceed the critical values;
  - a space ship lands on the planet and does not fly past it, etc.
- The only way to guarantee this is to have an interval $\mathbf{Y} = \left[\underline{Y}, \overline{Y}\right]$ for which $\mathbf{y} \subseteq \mathbf{Y}$ and $\overline{Y} \le y_0$.
- Such an interval is called an *enclosure*.
- Computing such an enclosure is one of the main tasks of interval computations.

22. Interval Computations: A Brief History

- *Origins:* Archimedes (Ancient Greece)
- *Modern pioneers:* Warmus (Poland), Sunaga (Japan), Moore (USA), 1956–59
- *First boom:* early 1960s.
- *First challenge:* taking interval uncertainty into account when planning spaceflights to the Moon.
- *Current applications* (sample):
  - design of elementary particle colliders: Berz, Kyoko (USA)
  - will a comet hit the Earth: Berz, Moore (USA)
  - robotics: Jaulin (France), Neumaier (Austria)
  - chemical engineering: Stadtherr (USA)

23. Interval Arithmetic: Foundations of Interval Techniques

- *Problem:* compute the range

  $$[\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) \,|\, x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

- *Interval arithmetic:* for arithmetic operations $f(x_1, x_2)$ (and for elementary functions), we have explicit formulas for the range.

- *Examples:* when $x_1 \in \mathbf{x}_1 = [\underline{x}_1, \overline{x}_1]$ and $x_2 \in \mathbf{x}_2 = [\underline{x}_2, \overline{x}_2]$, then:
  - The range $\mathbf{x}_1 + \mathbf{x}_2$ for $x_1 + x_2$ is $[\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2]$.
  - The range $\mathbf{x}_1 - \mathbf{x}_2$ for $x_1 - x_2$ is $[\underline{x}_1 - \overline{x}_2, \overline{x}_1 - \underline{x}_2]$.
  - The range $\mathbf{x}_1 \cdot \mathbf{x}_2$ for $x_1 \cdot x_2$ is $[\underline{y}, \overline{y}]$, where

    $$\underline{y} = \min(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \overline{x}_2, \overline{x}_1 \cdot \underline{x}_2, \overline{x}_1 \cdot \overline{x}_2);$$

    $$\overline{y} = \max(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \overline{x}_2, \overline{x}_1 \cdot \underline{x}_2, \overline{x}_1 \cdot \overline{x}_2).$$

- The range $1/\mathbf{x}_1$ for $1/x_1$ is $[1/\overline{x}_1, 1/\underline{x}_1]$ (if $0 \notin \mathbf{x}_1$).

24. Example

Let $\mathbf{x}_1 = [1, 2]$ and $\mathbf{x}_2 = [-3, 1]$. Then:

$$\mathbf{x}_1 + \mathbf{x}_2 = [1, 2] + [-3, 1] = [1 + (-3), 2 + 1] = [-2, 3];$$

$$\mathbf{x}_1 - \mathbf{x}_2 = [1, 2] - [-3, 1] = [1 - 1, 2 - (-3)] = [0, 5];$$

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = [1, 2] \cdot [-3, 1] =$$

$$[\min(1\cdot(-3), 1\cdot 1, 2\cdot(-3), 2\cdot 1), \max(1\cdot(-3), 1\cdot 1, 2\cdot(-3), 2\cdot 1)] =$$

$$[\min(-3, 1, -6, 2), \max(-3, 1, -6, 2)] = [-6, 2];$$

$$1/\mathbf{x}_1 = 1/[1, 2] = [1/2, 1/1] = [0.5, 1].$$

25. Exercise 4

- *Given:* $\mathbf{x}_1 = [-2, 3]$ and $\mathbf{x}_2 = [1, 4]$.
- *Task:* compute $\mathbf{x}_1 + \mathbf{x}_2$, $\mathbf{x}_1 - \mathbf{x}_2$, $\mathbf{x}_1 \cdot \mathbf{x}_2$, and $1/\mathbf{x}_2$.

26. Straightforward Interval Computations: Example

- *Example:* $f(x) = (x-2) \cdot (x+2)$, $x \in [1, 2]$.
- How will the computer compute it?
  - $r_1 := x - 2$;
  - $r_2 := x + 2$;
  - $r_3 := r_1 \cdot r_2$.
- *Main idea:* perform the same operations, but with *intervals* instead of *numbers*:
  - $\mathbf{r}_1 := [1, 2] - [2, 2] = [-1, 0]$;
  - $\mathbf{r}_2 := [1, 2] + [2, 2] = [3, 4]$;
  - $\mathbf{r}_3 := [-1, 0] \cdot [3, 4] = [-4, 0]$.
- *Actual range:* $f(\mathbf{x}) = [-3, 0]$.
- *Comment:* this is just a toy example, there are more efficient ways of computing an enclosure $\mathbf{Y} \supseteq \mathbf{y}$.

27. Exercise 5: Task

Use straightforward interval computations to compute the range of the function $f(x) = (x - 1) \cdot (x + 1)$ for $x \in [0, 2]$.

28. First Idea: Use of Monotonicity

- *Reminder:* for arithmetic, we had exact ranges.
- *Reason:* $+$, $-$, $\cdot$ are monotonic in each variable.
- *How monotonicity helps:* if $f(x_1, \ldots, x_n)$ is (non-strictly) increasing ($f \uparrow$) in each x_i, then
$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = [f(\underline{x}_1, \ldots, \underline{x}_n), f(\overline{x}_1, \ldots, \overline{x}_n)].$$
- *Similarly:* if $f \uparrow$ for some x_i and $f \downarrow$ for other x_j.
- *Fact:* $f \uparrow$ in x_i if $\dfrac{\partial f}{\partial x_i} \geq 0$.
- *Checking monotonicity:* check that the range $[\underline{r}_i, \overline{r}_i]$ of $\dfrac{\partial f}{\partial x_i}$ on $\mathbf{x}_i$ has $\underline{r}_i \geq 0$.
- *Differentiation:* by Automatic Differentiation (AD) tools.
- *Estimating ranges of* $\dfrac{\partial f}{\partial x_i}$*:* straightforward interval comp.

29. Monotonicity: Example

- *Idea:* if the range $[\underline{r}_i, \overline{r}_i]$ of each $\dfrac{\partial f}{\partial x_i}$ on $\mathbf{x}_i$ has $\underline{r}_i \geq 0$, then

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = [f(\underline{x}_1, \ldots, \underline{x}_n), f(\overline{x}_1, \ldots, \overline{x}_n)].$$

- *Example:* $f(x) = (x - 2) \cdot (x + 2)$, $\mathbf{x} = [1, 2]$.
- *Case $n = 1$:* if the range $[\underline{r}, \overline{r}]$ of $\dfrac{df}{dx}$ on $\mathbf{x}$ has $\underline{r} \geq 0$, then

$$f(\mathbf{x}) = [f(\underline{x}), f(\overline{x})].$$

- *AD:* $\dfrac{df}{dx} = 1 \cdot (x + 2) + (x - 2) \cdot 1 = 2x.$
- *Checking:* $[\underline{r}, \overline{r}] = [2, 4]$, with $2 \geq 0$.
- *Result:* $f([1, 2]) = [f(1), f(2)] = [-3, 0]$.
- *Comparison:* this is the exact range.

30. Exercise 6: Task

Use monotonicity to compute the range of the function $f(x) = (x-1)\cdot(x+1)$ for $x \in [0,2]$.

31. Non-Monotonic Example

- *Example:* $f(x) = x \cdot (1 - x)$, $x \in [0, 1]$.
- How will the computer compute it?
  - $r_1 := 1 - x$;
  - $r_2 := x \cdot r_1$.
- *Straightforward interval computations:*
  - $\mathbf{r}_1 := [1, 1] - [0, 1] = [0, 1]$;
  - $\mathbf{r}_2 := [0, 1] \cdot [0, 1] = [0, 1]$.
- *Actual range:* min, max of f at $\underline{x}$, $\overline{x}$, or when $\dfrac{df}{dx} = 0$.
- Here, $\dfrac{df}{dx} = 1 - 2x = 0$ for $x = 0.5$, thus we:
  - compute $f(0) = 0$, $f(0.5) = 0.25$, and $f(1) = 0$, so
  - $\underline{y} = \min(0, 0.25, 0) = 0$, $\overline{y} = \max(0, 0.25, 0) = 0.25$.
- *Resulting range:* $f(\mathbf{x}) = [0, 0.25]$.

32. Second Idea: Centered Form

- *Main idea:* Intermediate Value Theorem

$$f(x_1, \ldots, x_n) = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\chi) \cdot (x_i - \widetilde{x}_i)$$

  for some $\chi_i \in \mathbf{x}_i$.

- *Corollary:* $f(x_1, \ldots, x_n) \in \mathbf{Y}$, where

$$\mathbf{Y} = \widetilde{y} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \cdot [-\Delta_i, \Delta_i].$$

- *Differentiation:* by Automatic Differentiation (AD) tools.
- *Estimating the ranges of derivatives:*
  - if appropriate, by monotonicity, or
  - by straightforward interval computations, or
  - by centered form (more time but more accurate).

33. Centered Form: Example

- *General formula:*

$$\mathbf{Y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \cdot [-\Delta_i, \Delta_i].$$

- *Example:* $f(x) = x \cdot (1 - x)$, $\mathbf{x} = [0, 1]$.
- Here, $\mathbf{x} = [\widetilde{x} - \Delta, \widetilde{x} + \Delta]$, with $\widetilde{x} = 0.5$ and $\Delta = 0.5$.
- *Case $n = 1$:* $\mathbf{Y} = f(\widetilde{x}) + \dfrac{df}{dx}(\mathbf{x}) \cdot [-\Delta, \Delta]$.
- *AD:* $\dfrac{df}{dx} = 1 \cdot (1 - x) + x \cdot (-1) = 1 - 2x$.
- *Estimation:* we have $\dfrac{df}{dx}(\mathbf{x}) = 1 - 2 \cdot [0, 1] = [-1, 1]$.
- *Result:* $\mathbf{Y} = 0.5 \cdot (1 - 0.5) + [-1, 1] \cdot [-0.5, 0.5] = 0.25 + [-0.5, 0.5] = [-0.25, 0.75]$.
- *Comparison:* actual range $[0, 0.25]$, straightforward $[0, 1]$.

General Problem of . . .
Cauchy Deviates Method
Interval Arithmetic: . . .
Case Study: Chip Design
Combining Interval . . .
Case Study: . . .
Case Study: Detecting . . .
Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 34 of 80
Go Back
Full Screen
Close
Quit

34. Exercise 7: Task

Use centered form to compute the range of

$$f(x) = (x-1)\cdot(x+1)$$

for $x \in [-2, 4]$.

35. Third Idea: Bisection

- *Known:* accuracy $O(\Delta_i^2)$ of first order formula

$$f(x_1, \ldots, x_n) = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\chi) \cdot (x_i - \widetilde{x}_i).$$

- *Idea:* if the intervals are too wide, we:
  - split one of them in half ($\Delta_i^2 \to \Delta_i^2/4$); and
  - take the union of the resulting ranges.
- *Example:* $f(x) = x \cdot (1 - x)$, where $x \in \mathbf{x} = [0, 1]$.
- *Split:* take $\mathbf{x}' = [0, 0.5]$ and $\mathbf{x}'' = [0.5, 1]$.
- *1st range:* $1 - 2 \cdot \mathbf{x} = 1 - 2 \cdot [0, 0.5] = [0, 1]$, so $f \uparrow$ and $f(\mathbf{x}') = [f(0), f(0.5)] = [0, 0.25]$.
- *2nd range:* $1 - 2 \cdot \mathbf{x} = 1 - 2 \cdot [0.5, 1] = [-1, 0]$, so $f \downarrow$ and $f(\mathbf{x}'') = [f(1), f(0.5)] = [0, 0.25]$.
- *Result:* $f(\mathbf{x}') \cup f(\mathbf{x}'') = [0, 0.25]$ – exact.

36. Exercise 8: Task

Use bisection – and, if needed, monotonicity and centered form – to compute the range of $f(x) = (x-1)\cdot(x+1)$ for $x \in [-2, 4]$.

37. Alternative Approach: Affine Arithmetic

- *So far:* we compute the range of $x \cdot (1 - x)$ by multiplying ranges of x and $1 - x$.
- *We ignore:* that both factors depend on x and are, thus, dependent.
- *Idea:* for each intermediate result a, keep an explicit dependence on $\Delta x_i = \widetilde{x}_i - x_i$ (at least its linear terms).
- *Implementation:*

$$a = a_0 + \sum_{i=1}^{n} a_i \cdot \Delta x_i + [\underline{a}, \overline{a}].$$

- *We start:* with $x_i = \widetilde{x}_i - \Delta x_i$, i.e.,

$$\widetilde{x}_i + 0 \cdot \Delta x_1 + \ldots + 0 \cdot \Delta x_{i-1} + (-1) \cdot \Delta x_i + 0 \cdot \Delta x_{i+1} + \ldots + 0 \cdot \Delta x_n + [0, 0].$$

- *Description:* $a_0 = \widetilde{x}_i$, $a_i = -1$, $a_j = 0$ for $j \neq i$, and $[\underline{a}, \overline{a}] = [0, 0]$.

38. Affine Arithmetic: Operations

- *Representation:* $a = a_0 + \sum\limits_{i=1}^{n} a_i \cdot \Delta x_i + [\underline{a}, \overline{a}]$.
- *Input:* $a = a_0 + \sum\limits_{i=1}^{n} a_i \cdot \Delta x_i + \mathbf{a}$ and $b = b_0 + \sum\limits_{i=1}^{n} b_i \cdot \Delta x_i + \mathbf{b}$.
- *Operations:* $c = a \otimes b$.
- *Addition:* $c_0 = a_0 + b_0$, $c_i = a_i + b_i$, $\mathbf{c} = \mathbf{a} + \mathbf{b}$.
- *Subtraction:* $c_0 = a_0 - b_0$, $c_i = a_i - b_i$, $\mathbf{c} = \mathbf{a} - \mathbf{b}$.
- *Multiplication:* $c_0 = a_0 \cdot b_0$, $c_i = a_0 \cdot b_i + b_0 \cdot a_i$,

$$\mathbf{c} = a_0 \cdot \mathbf{b} + b_0 \cdot \mathbf{a} + \sum_{i \neq j} a_i \cdot b_j \cdot [-\Delta_i, \Delta_i] \cdot [-\Delta_j, \Delta_j] +$$

$$\sum_i a_i \cdot b_i \cdot [-\Delta_i, \Delta_i]^2 +$$

$$\left(\sum_i a_i \cdot [-\Delta_i, \Delta_i] \right) \cdot \mathbf{b} + \left(\sum_i b_i \cdot [-\Delta_i, \Delta_i] \right) \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b}.$$

39. Affine Arithmetic: Example

- *Example:* $f(x) = x \cdot (1 - x)$, $x \in [0, 1]$.
- Here, $n = 1$, $\widetilde{x} = 0.5$, and $\Delta = 0.5$.
- How will the computer compute it?
  - $r_1 := 1 - x$;
  - $r_2 := x \cdot r_1$.
- *Affine arithmetic:* we start with $x = 0.5 - \Delta x + [0, 0]$;
  - $\mathbf{r}_1 := 1 - (0.5 - \Delta x) = 0.5 + \Delta x$;
  - $\mathbf{r}_2 := (0.5 - \Delta x) \cdot (0.5 + \Delta x)$, i.e.,

$$\mathbf{r}_2 = 0.25 + 0 \cdot \Delta x - [-\Delta, \Delta]^2 = 0.25 + [-\Delta^2, 0].$$

- *Resulting range:* $\mathbf{y} = 0.25 + [-0.25, 0] = [0, 0.25]$.
- *Comparison:* this is the exact range.

40. Exercise 9: Task

Use affine arithmetic to compute the range of

$$f(x) = (x - 1) \cdot (x + 1)$$

for $x \in [-2, 4]$.

41. Affine Arithmetic: Towards More Accurate Estimates

- *In our simple example:* we got the exact range.
- *In general:* range estimation is NP-hard.
- *Meaning:* a feasible (polynomial-time) algorithm will sometimes lead to excess width: $\mathbf{Y} \supset \mathbf{y}$.
- *Conclusion:* affine arithmetic may lead to excess width.
- *Question:* how to get more accurate estimates?
- *First idea:* bisection.
- *Second idea* (Taylor arithmetic):
  - *affine arithmetic:* $a = a_0 + \sum a_i \cdot \Delta x_i + \mathbf{a}$;
  - *meaning:* we keep linear terms in Δx_i;
  - *idea:* keep, e.g., quadratic terms

$$a = a_0 + \sum a_i \cdot \Delta x_i + \sum a_{ij} \cdot \Delta x_i \cdot \Delta x_j + \mathbf{a}.$$

42. Interval Computations vs. Affine Arithmetic: Comparative Analysis

- *Objective:* we want a method that computes a reasonable estimate for the range in reasonable time.
- *Conclusion – how to compare different methods:*
  - how accurate are the estimates, and
  - how fast we can compute them.
- *Accuracy:* affine arithmetic leads to more accurate ranges.
- *Computation time:*
  - *Interval arithmetic:* for each intermediate result a, we compute two values: endpoints $\underline{a}$ and $\overline{a}$ of $[\underline{a}, \overline{a}]$.
  - *Affine arithmetic:* for each a, we compute $n + 3$ values:
$$a_0 \quad a_1, \ldots, a_n \quad \underline{a}, \overline{a}.$$
- *Conclusion:* affine arithmetic is $\sim n$ times slower.

43. Solving Systems of Equations: Extending Known Algorithms to Situations with Interval Uncertainty

- *We have:* a system of equations $g_i(y_1, \ldots, y_n) = a_i$ with unknowns y_i;
- *We know:* a_i with interval uncertainty: $a_i \in [\underline{a}_i, \overline{a}_i]$;
- *We want:* to find the corresponding ranges of y_j.
- *First case:* for exactly known a_i, we have an algorithm $y_j = f_j(a_1, \ldots, a_n)$ for solving the system.
- *Example:* system of linear equations.
- *Solution:* apply interval computations techniques to find the range $f_j([\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_n, \overline{a}_n])$.
- *Better solution:* for specific equations, we often already know which ideas work best.
- *Example:* linear equations $Ay = b$; y is monotonic in b.

44. Solving Systems of Equations When No Algorithm Is Known

- *Idea:*
  - parse each equation into elementary constraints, and
  - use interval computations to improve original ranges until we get a narrow range (= solution).
- *First example:* $x - x^2 = 0.5$, $x \in [0, 1]$ (no solution).
- *Parsing:* $r_1 = x^2$, $0.5\ (= r_2)\ = x - r_1$.
- *Rules:* from $r_1 = x^2$, we extract two rules:

$$(1)\ x \to r_1 = x^2; \quad (2)\ r_1 \to x = \sqrt{r_1};$$

from $0.5 = x - r_1$, we extract two more rules:

$$(3)\ x \to r_1 = x - 0.5; \quad (4)\ r_1 \to x = r_1 + 0.5.$$

45. Solving Systems of Equations When No Algorithm Is Known: Example

- (1) $r = x^2$; (2) $x = \sqrt{r}$; (3) $r = x - 0.5$; (4) $x = r + 0.5$.
- *We start with:* $\mathbf{x} = [0, 1]$, $\mathbf{r} = (-\infty, \infty)$.

(1) $\mathbf{r} = [0, 1]^2 = [0, 1]$, so $\mathbf{r}_{\rm new} = (-\infty, \infty) \cap [0, 1] = [0, 1]$.

(2) $\mathbf{x}_{\rm new} = \sqrt{[0, 1]} \cap [0, 1] = [0, 1]$ – no change.

(3) $\mathbf{r}_{\rm new} = ([0, 1] - 0.5) \cap [0, 1] = [-0.5, 0.5] \cap [0, 1] = [0, 0.5]$.

(4) $\mathbf{x}_{\rm new} = ([0, 0.5] + 0.5) \cap [0, 1] = [0.5, 1] \cap [0, 1] = [0.5, 1]$.

(1) $\mathbf{r}_{\rm new} = [0.5, 1]^2 \cap [0, 0.5] = [0.25, 0.5]$.

(2) $\mathbf{x}_{\rm new} = \sqrt{[0.25, 0.5]} \cap [0.5, 1] = [0.5, 0.71]$;
round $\underline{a}$ down $\downarrow$ and $\overline{a}$ up $\uparrow$, to guarantee enclosure.

(3) $\mathbf{r}_{\rm new} = ([0.5, 0.71] - 0.5) \cap [0.25, 5] = [0, 0.21] \cap [0.25, 0.5]$,
i.e., $\mathbf{r}_{\rm new} = \emptyset$.

- *Conclusion:* the original equation has no solutions.

46. Solving Systems of Equations: 2nd Example

- *Example:* $x - x^2 = 0$, $x \in [0, 1]$.
- *Parsing:* $r_1 = x^2$, $0\ (= r_2)\ = x - r_1$.
- *Rules:* (1) $r = x^2$; (2) $x = \sqrt{r}$; (3) $r = x$; (4) $x = r$.
- *We start with:* $\mathbf{x} = [0, 1]$, $\mathbf{r} = (-\infty, \infty)$.
- *Problem:* after Rule 1, we're stuck with $\mathbf{x} = \mathbf{r} = [0, 1]$.
- *Solution:* bisect $\mathbf{x} = [0, 1]$ into $[0, 0.5]$ and $[0.5, 1]$.
- *For 1st subinterval:*
  - Rule 1 leads to $\mathbf{r}_{\text{new}} = [0, 0.5]^2 \cap [0, 0.5] = [0, 0.25]$;
  - Rule 4 leads to $\mathbf{x}_{\text{new}} = [0, 0.25]$;
  - Rule 1 leads to $\mathbf{r}_{\text{new}} = [0, 0.25]^2 = [0, 0.0625]$;
  - Rule 4 leads to $\mathbf{x}_{\text{new}} = [0, 0.0625]$; etc.
  - we converge to $x = 0$.
- *For 2nd subinterval:* we converge to $x = 1$.

47. Exercise 10: Task

Use the above method to solve the system of equation

$$x + y = 1;$$

$$x - y = 0;$$

when $x \in [0, 0.8]$.

48. Optimization: Extending Known Algorithms to Situations with Interval Uncertainty

- *Problem:* find $y_1, \ldots, y_m$ for which

$$g(y_1, \ldots, y_m, a_1, \ldots, a_m) \to \max.$$

- *We know:* a_i with interval uncertainty: $a_i \in [\underline{a}_i, \overline{a}_i]$;
- *We want:* to find the corresponding ranges of y_j.
- *First case:* for exactly known a_i, we have an algorithm $y_j = f_j(a_1, \ldots, a_n)$ for solving the optimization problem.
- *Example:* quadratic objective function g.
- *Solution:* apply interval computations techniques to find the range $f_j([\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_n, \overline{a}_n])$.
- *Better solution:* for specific f, we often already know which ideas work best.

49. Optimization When No Algorithm Is Known

- *Idea:* divide the original box $\mathbf{x}$ into subboxes $\mathbf{b}$.
- If $\max\limits_{x\in\mathbf{b}} g(x) < g(x')$ for a known x', dismiss $\mathbf{b}$.
- *Example:* $g(x) = x\cdot(1-x)$, $\mathbf{x} = [0,1]$.
- Divide into 10 (?) subboxes $\mathbf{b} = [0, 0.1], [0.1, 0.2], \ldots$
- Find $g(\widetilde{b})$ for each $\mathbf{b}$; the largest is $0.45\cdot 0.55 = 0.2475$.
- Compute $G(\mathbf{b}) = g(\widetilde{b}) + (1 - 2\cdot\mathbf{b})\cdot[-\Delta, \Delta]$.
- Dismiss subboxes for which $\overline{Y} < 0.2475$.
- *Example:* for $[0.2, 0.3]$, we have

  $$0.25\cdot(1-0.25) + (1 - 2\cdot[0.2, 0.3])\cdot[-0.05, 0.05].$$
- Here $\overline{Y} = 0.2175 < 0.2475$, so we dismiss $[0.2, 0.3]$.
- *Result:* keep only boxes $\subseteq [0.3, 0.7]$.
- *Further subdivision:* get us closer and closer to $x = 0.5$.

50. Exercise 11: Task

Use this algorithm to find the minimum of the function $y = x^2$ on $[-1, 1]$. Divide into subboxes of width 0.4.

51. Case Study: Chip Design

- *Chip design:* one of the main objectives is to decrease the clock cycle.
- *Current approach:* uses worst-case (interval) techniques.
- *Problem:* the probability of the worst-case values is usually very small.
- *Result:* estimates are over-conservative – unnecessary over-design and under-performance of circuits.
- *Difficulty:* we only have *partial* information about the corresponding probability distributions.
- *Objective:* produce estimates valid for all distributions which are consistent with this information.
- *What we do:* provide such estimates for the clock time.

52. Estimating Clock Cycle: a Practical Problem

- *Objective:* estimate the clock cycle on the design stage.
- The clock cycle of a chip is constrained by the maximum path delay over all the circuit paths

$$D \stackrel{\text{def}}{=} \max(D_1, \ldots, D_N).$$

- The path delay D_i along the i-th path is the sum of the delays corresponding to the gates and wires along this path.
- Each of these delays, in turn, depends on several factors such as:
  - the variation caused by the current design practices,
  - environmental design characteristics (e.g., variations in temperature and in supply voltage), etc.

53. Traditional (Interval) Approach to Estimating the Clock Cycle

- *Traditional approach:* assume that each factor takes the worst possible value.
- *Result:* time delay when all the factors are at their worst.
- *Problem:*
  - different factors are usually independent;
  - combination of worst cases is improbable.
- *Computational result:* current estimates are 30% above the observed clock time.
- *Practical result:* the clock time is set too high – chips are over-designed and under-performing.

54. Robust Statistical Methods Are Needed

- *Ideal case:* we know probability distributions.
- *Solution:* Monte-Carlo simulations.
- *In practice:* we only have *partial* information about the distributions of some of the parameters; usually:
  - the mean, and
  - some characteristic of the deviation from the mean – e.g., the interval that is guaranteed to contain possible values of this parameter.
- *Possible approach:* Monte-Carlo with several possible distributions.
- *Problem:* no guarantee that the result is a valid bound for all possible distributions.
- *Objective:* provide *robust* bounds, i.e., bounds that work for all possible distributions.

55. Towards a Mathematical Formulation of the Problem

- *General case:* each gate delay d depends on the difference $x_1, \ldots, x_n$ between the actual and the nominal values of the parameters.
- *Main assumption:* these differences are usually small.
- Each path delay D_i is the sum of gate delays.
- *Conclusion:* D_i is a linear function: $D_i = a_i + \sum\limits_{j=1}^{n} a_{ij} \cdot x_j$ for some a_i and a_{ij}.
- The desired maximum delay $D = \max\limits_i D_i$ has the form

$$D = F(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \max_i \left(a_i + \sum_{j=1}^{n} a_{ij} \cdot x_j \right).$$

56. Towards a Mathematical Formulation of the Problem (cont-d)

- *Known:* maxima of linear function are exactly convex functions:

$$F(\alpha \cdot x + (1-\alpha) \cdot y) \le \alpha \cdot F(x) + (1-\alpha) \cdot F(y)$$

  for all x, y and for all $\alpha \in [0, 1]$;

- *We know:* factors x_i are independent;
  - we know distribution of some of the factors;
  - for others, we know ranges $[\underline{x}_j, \overline{x}_j]$ and means E_j.
- *Given:* a convex function $F \ge 0$ and a number $\varepsilon > 0$.
- *Objective:* find the smallest y_0 s.t. for all possible distributions, we have $y \le y_0$ with the probability $\ge 1-\varepsilon$.

57. Additional Property: Dependency is Non-Degenerate

- *Fact:* sometimes, we learn additional information about one of the factors x_j.
- *Example:* we learn that x_j actually belongs to a proper subinterval of the original interval $[\underline{x}_j, \overline{x}_j]$.
- *Consequence:* the class $\mathcal{P}$ of possible distributions is replaced with $\mathcal{P}' \subset \mathcal{P}$.
- *Result:* the new value y_0' can only decrease: $y_0' \le y_0$.
- *Fact:* if x_j is irrelevant for y, then $y_0' = y_0$.
- *Assumption:* irrelevant variables been weeded out.
- *Formalization:* if we narrow down one of the intervals $[\underline{x}_j, \overline{x}_j]$, the resulting value y_0 decreases: $y_0' < y_0$.

58. Formulation of the Problem

GIVEN:
- n, $k \le n$, $\varepsilon > 0$;
- a convex function $y = F(x_1, \ldots, x_n) \ge 0$;
- $n - k$ cdfs $F_j(x)$, $k + 1 \le j \le n$;
- intervals $\mathbf{x}_1, \ldots, \mathbf{x}_k$, values $E_1, \ldots, E_k$,

TAKE: all joint probability distributions on R^n for which:

- all x_i are independent,
- $x_j \in \mathbf{x}_j$, $E[x_j] = E_j$ for $j \le k$, and
- x_j have distribution $F_j(x)$ for $j > k$.

FIND: the smallest y_0 s.t. for all such distributions, $F(x_1, \ldots, x_n) \le y_0$ with probability $\ge 1 - \varepsilon$.

WHEN: the problem is *non-degenerate* – if we narrow down one of the intervals $\mathbf{x}_j$, y_0 decreases.

59. Main Result and How We Can Use It

- *Result:* y_0 is attained when for each j from 1 to k,
  - $x_j = \underline{x}_j$ with probability $\underline{p}_j \stackrel{\text{def}}{=} \dfrac{\overline{x}_j - E_j}{\overline{x}_j - \underline{x}_j}$, and
  - $x_j = \overline{x}_j$ with probability $\overline{p}_j \stackrel{\text{def}}{=} \dfrac{E_j - \underline{x}_j}{\overline{x}_j - \underline{x}_j}$.
- *Algorithm:*
  - simulate these distributions for x_j, $j < k$;
  - simulate known distributions for $j > k$;
  - use the simulated values $x_j^{(s)}$ to find

$$y^{(s)} = F(x_1^{(s)}, \ldots, x_n^{(s)});$$

  - sort N values $y^{(s)}$: $y_{(1)} \le y_{(2)} \le \ldots \le y_{(N_i)}$;
  - take $y_{(N_i \cdot (1-\varepsilon))}$ as y_0.

60. Comment about Monte-Carlo Techniques

- *Traditional belief:* Monte-Carlo methods are inferior to analytical:
  - they are approximate;
  - they require large computation time;
  - simulations for *several* distributions, may mis-calculate the (desired) maximum over *all* distributions.
- *We proved:* the value corresponding to the selected distributions indeed provide the desired maximum value y_0.
- *General comment:*
  - justified Monte-Carlo methods often lead to *faster* computations than analytical techniques;
  - example: multi-D integration – where Monte-Carlo methods were originally invented.

61. Comment about Non-Linear Terms

- *Reminder:* in the above formula $D_i = a_i + \sum\limits_{j=1}^{n} a_{ij} \cdot x_j$, we ignored quadratic and higher order terms in the dependence of each path time D_i on parameters x_j.
- *In reality:* we may need to take into account some quadratic terms.
- *Idea behind possible solution:* it is known that the max $D = \max\limits_i D_i$ of convex functions D_i is convex.
- *Condition when this idea works:* when each dependence $D_i(x_1, \ldots, x_k, \ldots)$ is still convex.
- *Solution:* in this case,
  - the function function D is still convex,
  - hence, our algorithm will work.

62. Conclusions

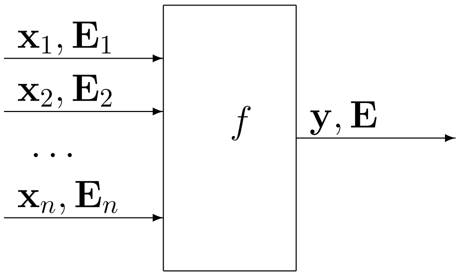
- *Problem of chip design:* decrease the clock cycle.
- *How this problem is solved now:* by using worst-case (interval) techniques.
- *Limitations of this solution:* the probability of the worst-case values is usually very small.
- *Consequence:* estimates are over-conservative, hence over-design and under-performance of circuits.
- *Objective:* find the clock time as y_0 s.t. for the actual delay y, we have $\text{Prob}(y > y_0) \le \varepsilon$ for given $\varepsilon > 0$.
- *Difficulty:* we only have *partial* information about the corresponding distributions.
- *What we have described:* a general technique that allows us, in particular, to compute y_0.

63. Combining Interval and Probabilistic Uncertainty: General Case

- *Problem:* there are many ways to represent a probability distribution.
- *Idea:* look for an objective.
- *Objective:* make decisions $E_x[u(x,a)] \to \max\limits_a$.
- *Case 1:* smooth $u(x)$.
- *Analysis:* we have $u(x) = u(x_0) + (x - x_0) \cdot u'(x_0) + \ldots$
- *Conclusion:* we must know moments to estimate $E[u]$.
- *Case of uncertainty:* interval bounds on moments.
- *Case 2:* threshold-type $u(x)$.
- *Conclusion:* we need cdf $F(x) = \text{Prob}(\xi \le x)$.
- *Case of uncertainty:* p-box $[\underline{F}(x), \overline{F}(x)]$.

64. Extension of Interval Arithmetic to Probabilistic Case: Successes

- *General solution:* parse to elementary operations $+$, $-$, $\cdot$, $1/x$, max, min.
- Explicit formulas for arithmetic operations known for intervals, for p-boxes $\mathbf{F}(x) = [\underline{F}(x), \overline{F}(x)]$, for intervals + 1st moments $E_i \stackrel{\text{def}}{=} E[x_i]$:



65. Successes (cont-d)

- *Easy cases:* $+$, $-$, product of independent x_i.
- *Example of a non-trivial case:* multiplication $y = x_1 \cdot x_2$, when we have no information about the correlation:
  - $\underline{E} = \max(p_1+p_2-1,0)\cdot\overline{x}_1\cdot\overline{x}_2+\min(p_1,1-p_2)\cdot\overline{x}_1\cdot\underline{x}_2+\min(1-p_1,p_2)\cdot\underline{x}_1\cdot\overline{x}_2+\max(1-p_1-p_2,0)\cdot\underline{x}_1\cdot\underline{x}_2$;
  - $\overline{E} = \min(p_1,p_2)\cdot\overline{x}_1\cdot\overline{x}_2+\max(p_1-p_2,0)\cdot\overline{x}_1\cdot\underline{x}_2+\max(p_2-p_1,0)\cdot\underline{x}_1\cdot\overline{x}_2+\min(1-p_1,1-p_2)\cdot\underline{x}_1\cdot\underline{x}_2$,

  where $p_i \stackrel{\text{def}}{=} (E_i - \underline{x}_i)/(\overline{x}_i - \underline{x}_i)$.

66. Example

- *Given:* $[\underline{x}_1, \overline{x}_1] = [\underline{x}_1, \overline{x}_2] = [-1, 1]$, $E_1 = E_2 = 0$.
- *Find:* the range of $E \stackrel{\text{def}}{=} E[x_1 \cdot x_2]$; solution:

$$p_1 = \frac{E_1 - \underline{x}_1}{\overline{x}_1 - \underline{x}_1} = \frac{0-(-1)}{1-(-1)} = \frac{1}{2} = 0.5,$$

- Similarly, $p_2 = 0.5$, so:

$$\underline{E} = \max(0.5+0.5-1,0)\cdot 1\cdot 1 + \min(0.5, 1-0.5)\cdot 1\cdot(-1) +$$
$$\min(1-0.5, 0.5)\cdot(-1)\cdot 1 +$$
$$\max(1-0.5-0.5, 0)\cdot(-1)\cdot(-1) = -0.5-0.5 = -1;$$
$$\overline{E} = \min(0.5, 0.5)\cdot 1\cdot 1 + \max(0.5-0.5, 0)\cdot 1\cdot(-1) +$$
$$\max(0.5-0.5, 0)\cdot(-1)\cdot 1 +$$
$$\min(1-0.5, 1-0.5)\cdot(-1)\cdot(-1) = 0.5+0.5 = 1.$$

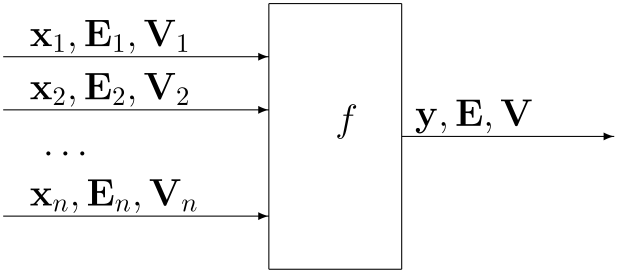
- *Solution:* the range is $[\underline{E}, \overline{E}] = [-1, 1]$.

67. Exercise 12: Task

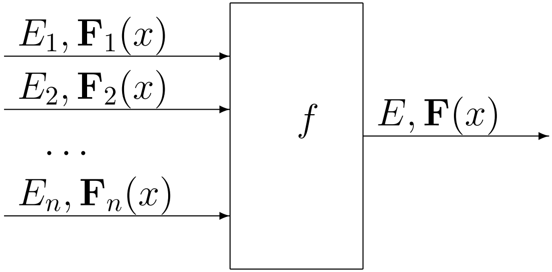
- *Given:* $[\underline{x}_1, \overline{x}_1] = [\underline{x}_1, \overline{x}_2] = [-1, 1]$, $E_1 = E_2 = 0.8$.
- *Find:* the range of $E \stackrel{\text{def}}{=} E[x_1 \cdot x_2]$.

68. Challenges

- intervals + 2nd moments:



- moments + p-boxes; e.g.:



General Problem of . . .
Cauchy Deviates Method
Interval Arithmetic: . . .
Case Study: Chip Design
Combining Interval . . .
Case Study: . . .
Case Study: Detecting . . .
Home Page
Title Page
Page 69 of 80
Go Back
Full Screen
Close
Quit

69. Case Study: Bioinformatics

- *Practical problem:* find genetic difference between cancer cells and healthy cells.
- *Ideal case:* we directly measure concentration c of the gene in cancer cells and h in healthy cells.
- *In reality:* difficult to separate.
- *Solution:* we measure $y_i \approx x_i \cdot c + (1 - x_i) \cdot h$, where x_i is the percentage of cancer cells in i-th sample.
- *Equivalent form:* $a \cdot x_i + h \approx y_i$, where $a \stackrel{\text{def}}{=} c - h$.

70. Case Study: Bioinformatics (cont-d)

- *If we know x_i exactly:* Least Squares Method $\sum_{i=1}^{n}(a \cdot x_i + h - y_i)^2 \to \min_{a,h}$, hence $a = \dfrac{C(x,y)}{V(x)}$ and $h = E(y) - a \cdot E(x)$, where $E(x) = \dfrac{1}{n} \cdot \sum_{i=1}^{n} x_i$,

$$V(x) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - E(x))^2,$$

$$C(x,y) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - E(x)) \cdot (y_i - E(y)).$$

- *Interval uncertainty:* experts manually count x_i, and only provide interval bounds $\mathbf{x}_i$, e.g., $x_i \in [0.7, 0.8]$.
- *Problem:* find the range of a and h corresponding to all possible values $x_i \in [\underline{x}_i, \overline{x}_i]$.

71. General Problem

- *General problem:*
  - we know intervals $\mathbf{x}_1 = [\underline{x}_1, \overline{x}_1], \ldots, \mathbf{x}_n = [\underline{x}_n, \overline{x}_n]$,
  - compute the range of $E(x) = \dfrac{1}{n}\sum\limits_{i=1}^{n} x_i$, population variance $V = \dfrac{1}{n}\sum\limits_{i=1}^{n}(x_i - E(x))^2$, etc.
- *Difficulty:* NP-hard even for variance.
- *Known:*
  - efficient algorithms for $\underline{V}$,
  - efficient algorithms for $\overline{V}$ and $C(x, y)$ for reasonable situations.
- *Bioinformatics case:* find intervals for $C(x, y)$ and for $V(x)$ and divide.

72. Case Study: Detecting Outliers

- In many application areas, it is important to detect *outliers*, i.e., unusual, abnormal values.
- In *medicine*, unusual values may indicate disease.
- In *geophysics*, abnormal values may indicate a mineral deposit (or an erroneous measurement result).
- In *structural integrity* testing, abnormal values may indicate faults in a structure.
- *Traditional engineering approach:* a new measurement result x is classified as an outlier if $x \notin [L, U]$, where

$$L \stackrel{\text{def}}{=} E - k_0 \cdot \sigma, \quad U \stackrel{\text{def}}{=} E + k_0 \cdot \sigma,$$

  and $k_0 > 1$ is pre-selected.
- *Comment:* most frequently, $k_0 = 2$, 3, or 6.

73. Outlier Detection Under Interval Uncertainty: A Problem

- In some practical situations, we only have intervals $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$.
- Different $x_i \in \mathbf{x}_i$ lead to different intervals $[L, U]$.
- A *possible* outlier: outside *some* k_0-sigma interval.
- *Example:* structural integrity – not to miss a fault.
- A *guaranteed* outlier: outside *all* k_0-sigma intervals.
- *Example:* before a surgery, we want to make sure that there is a micro-calcification.
- A value x is a possible outlier if $x \notin [\overline{L}, \underline{U}]$.
- A value x is a guaranteed outlier if $x \notin [\underline{L}, \overline{U}]$.
- *Conclusion:* to detect outliers, we must know the ranges of $L = E - k_0 \cdot \sigma$ and $U = E + k_0 \cdot \sigma$.

74. Outlier Detection Under Interval Uncertainty: A Solution

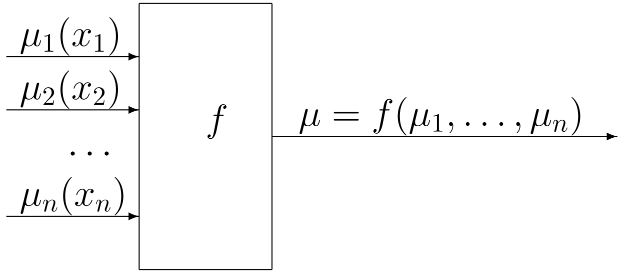
- *We need:* to detect outliers, we must compute the ranges of $L = E - k_0 \cdot \sigma$ and $U = E + k_0 \cdot \sigma$.
- *We know:* how to compute the ranges $\mathbf{E}$ and $[\underline{\sigma}, \overline{\sigma}]$ for E and σ.
- *Possibility:* use interval computations to conclude that $L \in \mathbf{E} - k_0 \cdot [\underline{\sigma}, \overline{\sigma}]$ and $L \in \mathbf{E} + k_0 \cdot [\underline{\sigma}, \overline{\sigma}]$.
- *Problem:* the resulting intervals for L and U are *wider* than the actual ranges.
- *Reason:* E and σ use the same inputs $x_1, \ldots, x_n$ and are hence not independent from each other.
- *Practical consequence:* we miss some outliers.
- *Desirable:* compute *exact* ranges for L and U.
- *Application:* detecting outliers in gravity measurements.

75. Acknowledgments

This work was supported in part by the National Science Foundation grants:

- HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and
- DUE-0926721.

76. Fuzzy Computations: A Problem



- *Given:* an algorithm $y = f(x_1, \dots, x_n)$ and n fuzzy numbers $\mu_i(x_i)$.
- *Compute:* $\mu(y) = \max\limits_{x_1,\dots,x_n: f(x_1,\dots,x_n)=y} \min(\mu_1(x_1), \dots, \mu_n(x_n))$.
- *Motivation:* y is a possible value of $Y \leftrightarrow \exists x_1, \dots, x_n$ s.t. each x_i is a possible value of X_i and $f(x_1, \dots, x_n) = y$.
- *Details:* "and" is min, $\exists$ ("or") is max, hence

$$\mu(y) = \max_{x_1,\dots,x_n} \min(\mu_1(x_1), \dots, \mu_n(x_n), t(f(x_1, \dots, x_n) = y)),$$

where $t(\text{true}) = 1$ and $t(\text{false}) = 0$.

77. Fuzzy Computations: Reduction to Interval Computations

- *Problem (reminder):*
  - *Given:* an algorithm $y = f(x_1, \ldots, x_n)$ and n fuzzy numbers X_i described by membership functions $\mu_i(x_i)$.
  - *Compute:* $Y = f(X_1, \ldots, X_n)$, where Y is defined by Zadeh's extension principle:

$$\mu(y) = \max_{x_1,\ldots,x_n : f(x_1,\ldots,x_n)=y} \min(\mu_1(x_1), \ldots, \mu_n(x_n)).$$

- *Idea:* represent each X_i by its α-cuts

$$X_i(\alpha) = \{x_i : \mu_i(x_i) \ge \alpha\}.$$

- *Advantage:* for continuous f, for every α, we have

$$Y(\alpha) = f(X_1(\alpha), \ldots, X_n(\alpha)).$$

- *Resulting algorithm:* for $\alpha = 0, 0.1, 0.2, \ldots, 1$ apply interval computations techniques to compute $Y(\alpha)$.

78. Proof of the Result about Chips

- Let us fix the optimal distributions for $x_2, \ldots, x_n$; then,

$$\mathrm{Prob}(D \le y_0) = \sum_{(x_1,\ldots,x_n): D(x_1,\ldots,x_n) \le y_0} p_1(x_1) \cdot p_2(x_2) \cdot \ldots$$

- So, $\mathrm{Prob}(D \le y_0) = \sum\limits_{i=0}^{N} c_i \cdot q_i$, where $q_i \stackrel{\text{def}}{=} p_1(v_i)$.
- Restrictions: $q_i \ge 0$, $\sum\limits_{i=0}^{N} q_i = 1$, and $\sum\limits_{i=0}^{N} q_i \cdot v_i = E_1$.
- Thus, the worst-case distribution for x_1 is a solution to the following linear programming (LP) problem:

Minimize $\sum\limits_{i=0}^{N} c_i \cdot q_i$ under the constraints $\sum\limits_{i=0}^{N} q_i = 1$ and

$$\sum_{i=0}^{N} q_i \cdot v_i = E_1,\ q_i \ge 0, \quad i = 0, 1, 2, \ldots, N.$$

79. Proof of the Result about Chips (cont-d)

- *Minimize:* $\sum\limits_{i=0}^{N} c_i \cdot q_i$ under the constraints $\sum\limits_{i=0}^{N} q_i = 1$ and $\sum\limits_{i=0}^{N} q_i \cdot v_i = E_1$, $q_i \geq 0$, $\;i = 0, 1, 2, \ldots, N$.
- *Known:* in LP with $N+1$ unknowns $q_0, q_1, \ldots, q_N$, $\geq N+1$ constraints are equalities.
- *In our case:* we have 2 equalities, so at least $N-1$ constraints $q_i \geq 0$ are equalities.
- Hence, no more than 2 values $q_i = p_1(v_i)$ are non-0.
- If corresponding v or v' are in $(\underline{x}_1, \overline{x}_1)$, then for $[v, v'] \subset \mathbf{x}_1$ we get the same y_0 – in contradiction to non-degeneracy.
- Thus, the worst-case distribution is located at $\underline{x}_1$ and $\overline{x}_1$.
- The condition that the mean of x_1 is E_1 leads to the desired formulas for $\underline{p}_1$ and $\overline{p}_1$.