Designing, Understanding, and Analyzing Unconventional Computation: The Important Role of Logic and Constructive Mathematics

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Main Problem: . . .
Our Main Claim: Use . . .
Why Logic?
1st Approach: . . .
1st Approach: Need . . .
Interval Computations
Proof Mining
2nd Approach: . . .
Potential Use of . . .
Potential Use of . . .
Explicit Use of . . .
Unconventional . . .

1. Main Problem: Reminder and Possible Approaches

- *Problem:* computations are often too slow.
- *Traditional approaches:*
  - design faster super-computers (*hardware*);
  - design faster *algorithms*.
- *Limitations of the traditional approaches:*
  - *re hardware:* we use the same physical processes as before;
  - *re algorithms:* we solve the same *exact* problem as before – while often, data are *imprecise*.
- *Possible new approaches:*
  - *hardware:* use unconventional physical (and biological) processes;
  - *algorithms:* perform computations only up to accuracy that matches the input accuracy.

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2. Our Main Claim: Use Logic (and Constructive Mathematics)

- *Alternative approaches (reminder):*
  - use unconventional physical (and biological) processes;
  - perform computations only up to accuracy that matches the input accuracy.
- *Our claim:* for these approaches to succeed, it is crucial to further develop:
  - the corresponding tools of mathematical *logic*, and
  - the related methods of *constructive mathematics.*

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3. Why Logic?

- *Claim:* logic is useful on all the stages of *solving a problem*:
    - we *specify* a problem;
    - we design and implement an *algorithm*;
    - we *verify* the corresponding program.
- *Specifying a problem:*
    - sometimes, the problem is to solve an equation;
    - in general, a proper formulation requires quantifiers etc. (stable control, etc.) – i.e., logic.
- *Designing an algorithm:* logic programming transforms a logical specification into an algorithm.
- *Program verification:* logic helps in reasoning about programs; e.g., pre-condition implies post-condition.
- *Proof assistant programs* (based on logic) help to prove.

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4. 1st Approach: Computations with Limited Accuracy

- *Objective:* compute $f(x)$ with accuracy $\varepsilon > 0$.
- *Idea:* compute x only with accuracy $\delta > 0$ for which

$$d(x, x') \leq \delta \text{ implies } d(f(x), f(x')) \leq \varepsilon.$$

- *Constructive math:* a *number* is r_n s.t. $d(r_n, x) \leq 2^{-n}$; a *constructive function* is a pair of:
  - an algorithm $f : X \to Y$ and
  - an algorithm $\varepsilon \to \delta$.
- *Status:* this is developed only for a few problems.
- *Research Direction I.1:*
  - *develop general constructive mathematics techniques,*
  - *with a special emphasis on problems requiring intensive computations (e.g., large-scale PDE).*

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5. 1st Approach: Need for Logic

- *Decomposition:* solutions to complex problems usually come from combining solutions to subproblems.
- *Logic is needed* for combination: e.g., stable robust control means it is stable $\forall$ possible parameter values.
- *Constructive logic:* we want to preserve constructions:
  - $\exists x\, P(x)$ should mean that we can construct such x;
  - $\forall x \exists y\, P(x,y) \leftrightarrow \exists$ algorithm $\varphi : X \to Y$ s.t. $P(x, \varphi(x))$.
- It was originated by Kolmorogov: e.g., $A \vee \neg A$ is false.
- *Fact:* constructive logic is used in constructive math.
- *Research Direction I.2:*
  - *develop general constructive logic techniques,*
  - *with a special emphasis on problems requiring intensive computations.*

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6. Interval Computations

- *Constructive math:* algorithms work for all accuracies.
- *In practice:* we only need one given accuracy.
- *Solution:* interval computations (a.k.a. applied constructive mathematics).
- *Idea:* if we know an estimate $\widetilde{x}$ w/accuracy Δ, then

$$x \in \mathbf{x} = [\widetilde{x} - \Delta, \widetilde{x} + \Delta].$$

- *Traditional approach:* we also know probability distribution for $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$ (usually Gaussian).
- *Where it comes from:* calibration using standard MI.
- *Problem:* calibration is not possible in:
  - fundamental science – no better Measuring Instr. (MI);
  - manufacturing – too expensive to calibrate.

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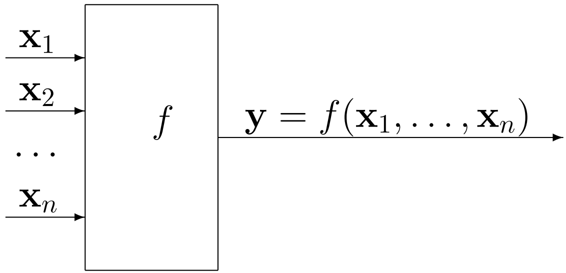
7. Interval Computations (cont-d)



- *Given:* algorithm $y = f(x_1, \ldots, x_n)$ and $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$.
- *Compute:* the corresponding range of y:

$$\mathbf{y} = [\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) \mid x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

- *Fact:* this problem is NP-hard even for quadratic f.
- *Challenge:* find a good approximation $\mathbf{Y} \supseteq \mathbf{y}$.
- *Applications:* spaceflights, super-colliders, robotics, chemical engineering, nuclear safety, etc.
- *Modal logic* is efficiently used: $\Box\,(y \in \mathbf{y})$, $\Diamond\,(x = \widetilde{x})$.

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8. Proof Mining

- *Historically:* first existence proofs were *direct* (constr.).
- *Currently:* many proofs are *indirect* – they prove $\exists x\, P(x)$ without constructing such x.
- *Meta-results:* sometimes, we can extract ("mine") a constructive proof from a non-constructive one.
- *Example* (Kohlenbach): uniqueness $\to$ computability.
- *Idea:* to find x s.t. $f(x) = 0$, compute $\min_{B_\varepsilon(x_i)} |f(x)|$ w/increasing accuracy for x_i from ε-nets, $\varepsilon = 2^{-1}, 2^{-2}, \ldots$
- *Research Direction I.4:*
  - *further develop proof mining,*
  - *with a special emphasis on its use to develop algorithms for realistic large-scale problems.*

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9. 2nd Approach: Unconventional Computations

- *Main idea:* use non-standard physical processes to speed up computations.
- *Most well-known example:* quantum computing:
  - search in an un-sorted array of size n in time $\sqrt{n}$ (Grover);
  - factoring large integers in polynomial time (Shor).
- *Limitation:* the only provable speed-up is polynomial.
- *Other schemes:* can potentially lead to exponential speed-up.
- *In other words:* we can potentially solve NP-hard problems in polynomial time.
- *Simplest example:* acausal processes – compute and send the result back in time.

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10. Potential Use of Acausal Processes (cont-d)

- *Idea* (reminder): compute and send the result back in time.
- *Problem:* paradoxes of time travel – e.g., killing your own grandfather before your father was conceived.
- *Solution:* some low probability event prevented the time traveller from this killing.
- *Consequence:* time travel (TT) can trigger events with probability $p_0 \ll 1$.
- *Typical NP-hard problem:* SAT – given a propositional formula $F(x)$, find $x = (x_1, \ldots, x_n)$ s.t. $F(x)$ holds.
- *Usage* (H. Moravec et al.): to solve SAT, generate n bits x and if $\neg F(x)$, launch TT.
- *Why it works:* for $2^{-n} \gg p_0$, TT is statistically improbable.

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11. Potential Use of Curved Space-Time

- *Parallelization* – a natural source of speed-up.
- *Claim:* in Euclidean space-time, parallelization only leads to a polynomial speed-up.
- *Fact:* the speed of all the physical processes is bounded by the speed of light c.
- *Conclusion:* in time T, we can only reach computational units at a distance $\leq R = c \cdot T$.
- The volume $V(R)$ of this area (inside of the sphere of radius $R = c \cdot T$) is proportional to $R^3 \sim T^3$.
- So, we can use $\leq V/\Delta V \sim T^3$ computational elements.
- *Interesting:* in Lobachevsky space-time, $V(R) \sim \exp(R)$.
- Same is true for some more realistic space-time models.
- Hence, we can fit exponentially many processors – and thus get an exponential speed-up.

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12. Explicit Use of Kolmogorov Complexity

- *Fact:* it's often difficult to describe biological processes.
- *Idea* (M. Gell-Mann): physical equations should include terms explicitly depending on complexity.
- *Natural formalization:* Kolmogorov complexity

$$K(x) \stackrel{\text{def}}{=} \min\{\text{len}(p) : p \text{ generates } x\}.$$

- *Conclusion:* by observing physical and biological processes, we can measure the value $K(x)$.
- *Observation:* $K(x)$ is not algorithmically computable.
- *Known results:* ability to get non-computable values can speed up computations.
- *Other schemes* are based on:
  - quantum field theory (G. Kreisel),
  - that every theory is approximate, etc.

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13. Unconventional Computations (UC) and Constructive Mathematics

- *Above UC schemes:* use or propose a radically *new* physical process.
- *Fact:* some UC schemes were discovered by analyzing computability of *known* physical equations.
- *Example* (M. Pour-El et al.): even for wave equation, for some computable $u(x, 0)$, $u(x, T)$ is not computable.
- *Desirable:* extend the existing UC activity to the analysis of what computations can be sped up.
- *Research Direction II.1. Use constructive mathematics to analyze:*
  - *how the use of physical processes (described by physically meaningful equations)*
  - *can speed up computations.*

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