Data Processing in the Presence of Interval Uncertainty and Erroneous Measurements: Practical Problems, Results, Challenges

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1. Formulation of the Problem

- There are *two main reasons* why measurement results differ from the actual values of the measured quantities:
- There is a *small* difference caused by the inaccuracy of the measuring instrument.
- This inaccuracy is characterized by *probabilistic* or *interval* uncertainty.
- Sometimes, due to an instrument malfunction or a human error, we get an erroneous measurement result (outlier) that is *drastically* different from the actual value.
- This uncertainty is usually characterized by a *proportion* of measurement results that could be erroneous (e.g., $\leq 1\%$).
- *Situation:* most data processing algorithms based on interval computations only take into account the first type of uncertainty.
- *Problem:* take the presence of erroneous measurements into account as well.

2. Sometimes, It Is Relatively Easy to Detect Outliers

- In some cases, when the data is smooth, we can (rather easily) detect the corresponding outliers.
- *Traditional engineering approach:* a new measurement result x is classified as an outlier if $x \notin [L, U]$, where

$$L \stackrel{\text{def}}{=} E - k_0 \cdot \sigma, \quad U \stackrel{\text{def}}{=} E + k_0 \cdot \sigma,$$

  and $k_0 > 1$ is pre-selected (most frequently, $k_0 = 2$, 3, or 6.
- *Minor problem:* in some practical situations, we only have intervals $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$.
- For different values $x_i \in \mathbf{x}_i$, we get different k_0-sigma intervals $[L, U]$.
- A value x is a guaranteed outlier if $x \notin [\underline{L}, \overline{U}]$.
- *Conclusion:* to detect outliers, we must know the ranges of $L = E - k_0 \cdot \sigma$ and $U = E + k_0 \cdot \sigma$.
- *Good news:* there exist algorithm for computing these ranges.
- *Not so good news:* in many practical situations, e.g., in non-destructive testing (NDT) of aeroplanes and roads, and in geophysical analysis, we are actually interested in unusual non-smooth data points.
- *Problem:* separating correct but unusual measurement results from the erroneous measurement results is a challenge.

3. Presence of Erroneous Measurements Make Problems Computationally Difficult

- *Known fact:* the presence of outliers turns easy-to-solve interval problems into difficult-to-solve (NP-hard) ones.
- *New result:* this difficulty may appear even without interval uncertainty.
- *Situation:* we know how the measured quantity y is related to the desired parameters x_j.
- *Simplest case:* linear dependence, i.e., $\sum\limits_{j=1}^{n} a_{ij} \cdot x_j = y_i$, where y_i is the result of i-th measurement, and a_{ij} are (known) parameters corresponding to i-th measurement.
- *Problem:* given a_{ij}, y_i, and $\varepsilon \in (0, 1)$, and constraints

$$\sum_{j=1}^{n} a_{ij} \cdot x_j = y_i, \quad i = 1, \ldots, N$$

check whether we can select a consistent set of $N \cdot (1 - \varepsilon)$ constraints.

4. Result: The Problem Is NP-hard Even for the Linear Case

- *Idea:* reduce to a known NP-hard problem.
- *Subset sum:* given positive integers $s_1, \ldots, s_n$, and s, check whether $s = \sum_{i=1}^{n} x_i \cdot s_i$ for some $x_i \in \{0, 1\}$.
- *Reduction:* $N = n/\varepsilon$ constraints:
  - $2n$ constraints $x_1 = 0$, $x_1 = 1 \ldots$, $x_n = 0$, $x_n = 1$;
  - $N - 2n$ identical constraints $\sum s_i \cdot x_i = s$.
- Since $0 \neq 1$, at most $N - n$ are satisfied.
- If the subset problem has a solution, then:
  - all $N - 2n$ constraints $\sum s_i \cdot x_i = s$ are satisfied,
  - and for each i, $x_i = 0$ or $x_i = 1$,

  to the total of $N - n = N \cdot (1 - \varepsilon)$.
- If $N - n$ constraints are satisfied, then for every i, $x_i \in \{0, 1\}$ – a solution to the subset problem.

5. Constraint Propagation Techniques (Semenov, Numerica, Jaulin, etc): Reminder

- Constraint propagation – traditional technique for solving constraint satisfaction problems.
- We start with the intervals $[\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]$ containing the actual values of the unknowns $x_1, \ldots, x_n$.
- On each iteration:
  - select i and a constraint $f_j(x_1, \ldots, x_n) = 0$,
  - replace $[\underline{x}_i, \overline{x}_i]$ with new interval $\mathbf{x}_i^{(j)} = [\underline{x}_i^{(j)}, \overline{x}_i^{(j)}] \stackrel{\text{def}}{=}$

$$\{x_i : x_i \in [\underline{x}_i, \overline{x}_i] \,\&\, f_j(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = 0$$
$$\text{for some } x_k \in [\underline{x}_k, \overline{x}_k]\}.$$

- If the process stalls, we bisect the interval for one the variables into two and try to decrease both resulting half-boxes.
- *Problem:* cannot use it if not all constraints are valid.

6. Traditional Interval-Related Constraint Propagation Techniques: Example

- *Toy problem:* find $x \in [-5, 5]$ for which $x - x^2 = 0$.
- *Pre-processing:* parse the expression:

$$r = x^2; x - r = 0.$$

- *Originally:* $X = [-5, 5]$, $R = [-\infty, \infty]$.
- *Use the first constraint:* $x \in [-5, 5]$ implies $r \in [0, 25]$, so for r, the new interval is $[-\infty, \infty] \cap [0, 25] = [0, 25]$:

$$X = [-5, 5], \quad R = [0, 25].$$

- *Use the second constraint:* for x, we have $[-5, 5] \cap [0, 25] = [0, 5]$, and similarly for r, so

$$X = [0, 5], \quad R = [0, 5].$$

- *Use the first constraint:* $x = \sqrt{r}$, hence

$$X = [0, 2.24], \quad R = [0, 5].$$

- *Use the second constraint:*

$$X = [0, 2.24], \quad R = [0, 2.24].$$

- After a while, we stall at $X = R \approx [0, 1]$, so we bisect X to $[0, 1/2]$ and $[1/2, 1]$.
- Then, we converge to $x = 0$ and $x = 1$.

7. New Idea

- On each iteration, we still select a variable x_i, but:
  - instead of selecting a *single* constraint,
  - we try *all* N constraints, and get N resulting intervals $[\underline{x}_i^{(j)}, \overline{x}_i^{(j)}]$.
- We know that $\geq N \cdot (1-\varepsilon)$ constraints are satisfied.
- Hence $x_i \leq \overline{x}_i^{(j)}$ for $\geq N \cdot (1-\varepsilon)$ different values j.
- Let us sort all N upper endpoints $\overline{x}_i^{(j)}$ $(1 \leq j \leq N)$ into an increasing sequence $u_1 \leq u_2 \leq \ldots \leq u_N$,
- Then we can guarantee that x_i is smaller than (or equal to) at least $N \cdot (1-\varepsilon)$ terms in this sequence.
- So, $x_i \leq u_{N \cdot \varepsilon}$.
- Similarly, if we sort the lower endpoints $\underline{x}_i^{(j)}$ into a decreasing sequence $l_1 \geq \ldots \geq l_N$, then $x_i \geq l_{N \cdot \varepsilon}$.

8. New Algorithm

- On each iteration, we:
  - we select a variable x_i;
  - for each of N constraints, we compute the corresponding interval $[\underline{x}_i^{(j)}, \overline{x}_i^{(j)}]$;
  - we sort all N upper endpoints $\overline{x}_i^{(j)}$ $(1 \le j \le N)$ into an increasing sequence $u_1 \le u_2 \le \ldots \le u_N$,
  - we sort all N lower endpoints $\underline{x}_i^{(j)}$ $(1 \le j \le N)$ into a decreasing sequence $l_1 \ge l_2 \ge \ldots \ge l_N$, and
  - we take $[l_{N \cdot \varepsilon}, u_{N \cdot \varepsilon}]$ as the new interval for x_i.
- If the process stalls, we bisect the interval and try to decrease both resulting half-boxes.
- *Comment:* producing $u_{N \cdot \varepsilon}$ can be done faster than by sorting.

9. Other Potential Applications of the New Algorithm: Design and Control Problems

- In many areas of science and engineering, we are interested in solving *design* and *control* problems.
- *In mathematical terms:* a design or a control can be usually represented by the values of the relevant numerical parameters $x = (x_1, \ldots, x_n)$.
- Usually, in these problems, the users describe several *constraints* that the desired design or control must satisfy.
- *Objective:* find a design (corr., a control) that satisfies all these constraints.

10. How to Describe Constraints?

- *Example:* an airplane design can be described in terms of:
  - the geometric parameters of the plane,
  - the thickness of the plates that form the airplane's skin,
  - the weight and power of the engine, etc.
- *Typical constraint:* a limitation on some characteristics $y = f(x_1, \ldots, x_n)$ of this design.
- *Examples*
  - the airplane's speed must exceed some y_0,
  - its fuel use must not exceed a certain amount,
  - the overall cost must be within given limits.
- So, constraints are of the type $f(x_1, \ldots, x_n) \le y_0$ or $f(x_1, \ldots, x_n) \ge y_0$ (or $f(x_1, \ldots, x_n) = y_0$).

11. Constraint Satisfaction vs. Constrained Optimization

- *Constraint satisfaction:* find a design that satisfies given constraints.
- *Problem:*
  - different designs that satisfy the given constraints;
  - we must select one of these designs.
- Users can often describe their preference in terms of an *objective function* $g(x_1, \ldots, x_n)$ (whose value should be made as large as possible).
- *Constrained optimization:* maximizing $g(x_1, \ldots, x_n)$ under the given constraints.
- *In general:* both problem are NP-hard.
- *In practice:* there are many efficient tools for solving them.

12. "Soft" Constraints

- *Problem:* sometimes, the users constraints are inconsistent.
- *Example:* design a plane that is:
  - as fast and as fuel-efficient as the existing Airbus or Boeing planes,
  - but with 0 noise level.
- *Reasons for inconsistency:*
  - some constraints are *absolute* (e.g., safety constraints),
  - others are *desires* – they can be dismissed if not possible.
- Such "not required" constraints are called *soft constraints.*
- *Comment:* soft constraints are an important research topic, with annual conferences.
- *Idea:* when we cannot satisfy *all* the constraints, we should satisfy as *many* constraints as possible.

13. Case Study: Seismic Inverse Problem

- *Problem:* to determine the geophysical structure of a region.
- *Solution:* we:
  - measure seismic travel times, and
  - reconstruct velocities at different depths from this data.
- *Difficulty:* the inverse problem is ill-defined:
  - *large* changes in the original distribution of velocities can lead to
  - very *small* changes in the resulting measured values.
- *Conclusion:* many different velocity distributions are consistent with the same measurement results.

14. Drawbacks of the Existing Approach

- *Situation:* because of the non-uniqueness, the velocity distribution that is returned by the existing algorithm is usually not geophysically meaningful.
- *Example:* it predicts velocities outside of the range of reasonable velocities at this depth.
- *Current solution:* a geophysicist adjusts the initial approximation so as to avoid this discrepancy.
- *Problem:* several iterations are needed; it is very time-consuming.
- *Problem:* adjustment requires special difficult-to-learn skills.
- *Result:* the existing tools for solving the seismic inverse problem are not as widely used as they could be.

15. It Is Necessary to Take Expert Knowledge Into Consideration

- *Objective:* make the tools for processing seismic data more accessible.
- *Solution:* incorporate the expert knowledge into the algorithm for solving the inverse problem.
- *Example why expert knowledge is needed:* velocity is outside the interval of values which are possible at this depth for this particular geological region.
- *Corresponding expert knowledge:* the intervals of possible values of data.
- *What needs to be done:* modify the inverse algorithms in such a way that the velocities are always within these intervals.
- *Question:* how can we do it?

16. How We Can Use Interval Uncertainty

- *How algorithms work now:*
  - start with a reasonable velocity model;
  - predict traveltimes x_i between stations;
  - use the difference $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$, where $\widetilde{x}_i$ are measured values, to adjust the velocity model:
    - divide Δx_i by the length L of the path;
    - add $\Delta x_i / L$ to all slownesses along the path.
- *How to modify* when we know the interval $[\underline{s}_j, \overline{s}_j]$ of possible slownesses:
  - first, we compute the next approximation $s_j^{(k)}$ to the slownesses,
  - then, we replace $s_j^{(k)}$ with the nearest value within the interval $[\underline{s}_j, \overline{s}_j]$.

17. Explicit Expert Knowledge: Fuzzy Uncertainty

- Experts can usually produce a wider interval of which they are practically 100% certain.
- In addition, experts can also produce narrower intervals about which their degree of certainty is smaller.
- As a result, instead of a *single* interval, we have a *nested* family of intervals corresponding to different levels of uncertainty.
- In effect, we get a *fuzzy* interval (of which different intervals are α-cuts).
- *Previously:* a solution is satisfying or not.
- *New idea:* a satisfaction *degree* d.
- *Specifics:* d is the largest α for which all s_i are within the corresponding α-cut intervals.

18. Implicit Expert Knowledge: Interval Uncertainty

- *Situation:* sometimes, velocities are in the interval, but the geophysical structure is still not right.
- *Explanation:*
  - algorithms assume that the measured errors are independent and normally distributed;
  - so, stopping criterion is MSE $E \stackrel{\text{def}}{=} \sum\limits_{i=1}^{N} (x_i - \widetilde{x}_i)^2$;
  - for geophysically meaningless models, E is small, but some differences $x_i - \widetilde{x}_i$ are large.
- *Solution:* require that $|x_i - \widetilde{x}_i| \leq \Delta$ for some bound Δ.

19. How We Can Use Interval Uncertainty

- *Problem:* how can we guarantee that we only get solutions which are physical in the above sense?
- *Traditional approach:* once the mean square error is small, we stop iterations.
- *Natural new idea:* continue iterations until *all* (or rather almost all, with proportion $\geq 1 - \varepsilon$) differences $|x_i - \widetilde{x}_i|$ are under Δ.
- *Question:* what if this does not happen?
- *Similar question:* what traditional algorithms do if we do not MSE small?
- *Answer to similar question:* restart computations with a different starting velocity model.
- *Solution to our problem:* restart computations with a different starting velocity model.

20. A General Problem

- Inverse problem is ill-posed $\approx$ has many different solutions.
- Many inverse problems in science and engineering are ill-posed.
- *Regularization:* we select a solution with a certain property, e.g., a smooth one, $J \stackrel{\text{def}}{=} \int (x'(t))^2\,dt \to \min$.
- *Discrete case:* $J_{\text{discr}} \stackrel{\text{def}}{=} \sum\limits_i (x(t_{i+1}) - x(t_i))^2$.
- *2-D case:* $J \stackrel{\text{def}}{=}$
$$\sum_{n_1,n_2} [(f(n_1+1,n_2) - f(n_1,n_2))^2 + (f(n_1,n_2+1) - f(n_1,n_2))^2],$$
or, equivalently, $J = \sum\limits_{p,p' \text{ are neighbors}} (f(p) - f(p'))^2$.
- Smoothness leads to efficient algorithms.
- *Problem:* for inverse problem in geophysics, we only have piecewise smoothness.

21. General Problem: Precise Formulation

- *Idea:* we only take into account the pairs of neighboring pixels that belong to the same zone:
$$J(Z) = \sum_{p,p' \text{ are neighbors in the same zone}} (f(p) - f(p'))^2,$$
where Z denotes the information about the zones.
- Often, we do not know where the edges are, i.e., we do not know Z.
- *Idea:* find Z for which the result inside each zone is the smoothest, i.e., minimize
$$J^* = \min_{\text{all possible divisions } Z \text{ into zones}} J(Z).$$
- *Problem:* the resulting problem is no longer convex.
- It is known that non-convex problems are, in general, more computationally difficult.

22. Result: Reconstructing Piecewise Smooth Solutions is NP-Hard

- *Idea of the proof:* we reduce a known NP-hard problem (subset sum) to our problem.
- *Subset sum:*
  - given m positive integers $s_1, \ldots, s_m$ and an integer $s > 0$,
  - check whether it is possible to find a subset of this set of integers whose sum is equal to exactly s.
- *Alternative description:* check whether there exist $x_i \in \{0, 1\}$ for which $\sum s_i \cdot x_i = s$.

23. Reduction

- We want to reconstruct an $m \times m$ solution $f(n_1, n_2)$.
- Let $d = \lfloor m/2 \rfloor$. We want a piecewise smooth solution $f(n_1, n_2)$ that consists of two zones.
- The following linear constraints describe the consistency between the observations and the desired solution:
  - $f(n_1, n_2) = 1$ for $n_2 > d$;
  - $\sum\limits_{i=1}^{m} s_i \cdot f(i, d) = s$; and
  - $f(n_1, n_2) = 0$ for $n_2 < d$.
- *Problem:* among all the solutions that satisfy these constraints, find the one with the smallest non-smoothness J^*.

24. Proof

- Let us show that $\min J^* = 0 \leftrightarrow$ the original subset problem has a solution.
- If $J^* = 0$, then all the values within each zone must be the same.
- Since $f = 1$ for $n_2 > d$ and $f = 0$ for $n_2 < d$, every value $f(n_1, n_2)$ is $= 1$ or $= 0$.
- Thus, the values $x_i = f(i, d) \in \{0, 1\}$ solve the original subset problem $\sum s_i \cdot x_i = s$.
- Vice versa:
  - if the selected instance of the original subset problem has a solution x_i,
  - then we can take $f(i, d) = x_i$ and get the solution of the inverse problem for which the degree of non-smoothness is exactly 0.

25. Acknowledgments

This work was supported in part:

- by NASA under cooperative agreement NCC5-209,
- by NSF grants EAR-0112968, EAR-0225670, and EIA-0321328, and
- by NIH grant 3T34GM008048-20S1.

26. In Many Practical Problems, It Is Very Important to Test Smoothness

- In many practical problems, we must check whether a given object is smooth or whether it has non-smooth areas:
  - *aerospace structure:* cracks, holes, other faults;
  - *mammography:* small clots, cracks, etc., which may indicate a tumor.
- *Smoothness leads to linearity*
  - If a tested structure has no faults, then the surface is usually *smooth.*
  - As a result, the dependencies f_i between the test signals x_j and the received signals y_i are also smooth.
  - Since we are sending relatively weak signals x_i (strong signals can damage the plane), we can neglect quadratic (and higher order) terms in Taylor series and only consider linear terms in these series.
- *Non-smoothness leads to non-linearity*
  - A fault (e.g., a crack) is, usually, a violation of smoothness.
  - Thus, if there is a fault, the structure stops being smooth; hence, the function f_j stops being smooth.
  - Therefore, linear terms are no longer sufficient.
- So, we can detect the fault by checking whether the dependency between y_j and x_i is linear.

27. The Resulting Proposal: Main Idea

As a result of the above analysis, we propose the following way of detecting faults:

- We apply different signals x_j to the object, and measure the response y_i.
- If the measurement results are consistent with the *linear* dependence of y_i on x_j, this means that there are no faults, and no further testing is needed.
- If the measurement results are inconsistent with the linear model, this means that there is a fault, and so further thorough tests are needed.

This proposal saves time and resources:

- Checking linearity is easy.
- As a result, for non-destructive evaluation of aerospace structures, we get a simple test that:
  - enables us to *save* time and resources (necessary for the detailed solution of the inverse problem)
  - by *limiting* this detalization only to the cases when the presence of the faults was revealed by non-linearity.

28. Mechanical Analysis of the Problem

- Fault-less plate:
  - Transmitter sends a signal $x(t) = A \cdot \cos(\omega \cdot t)$.
  - This signal travels to the receiver (at a distance d) with a speed of sound v, and thus gets delayed by $\Delta t = d/v$.
  - Hence, the received signal is

    $$y(t) = k \cdot x(t - \Delta t) = k \cdot A \cdot \cos(\omega \cdot t - \omega \cdot d/v),$$

    where the coefficient k describes the loss of amplitude.

  Thus, for a fault-less plate, we indeed have a linear dependence between the transmitted signal $x(t)$ and the measured signal $y(t)$.

- Plate with faults:
  - For a plate with a crack, $\Delta t = d/v + d_0/v_0$, where d_0 is the linear size of the fault, and v_0 is the speed of sound in the air.
  - As we transmit the signal $x(t)$, the plate starts vibrating.
  - This vibration changes the position of both borders of the crack and therefore, changes (harmonically) the distance d_0: $d_0 = d_0(t)$.
  - So, we get a non-linear phase:

    $$y(t) = k \cdot A \cdot \cos(\omega \cdot t - \omega \cdot d/v - \omega \cdot d_0(t)/v_0),$$

    and hence a non-linear dependence.

29. Experimental Confirmation of Non-Linearity: Pseudo-Random Signals

For *pseudo-random signals* $x(t)$ (which combine components of several different frequencies with pseudo-random amplitudes and pseudo-random phases):

- For a fault-less plate, the dependence between the transmitted signal $x(t)$ and the measured signal $y(t)$ is linear, i.e.,

$$y(t) = \int A(t-s) \cdot x(s)\, ds$$

  for some function $A(t)$.

- For a plate with a fault, this dependence is non-linear: namely, cubic terms must be taken into consideration.

The amplitude of the cubic term is roughly proportional to the cube of the linear fault size. Thus:

- not only the non-linear terms indicate the *presence* of the fault, but also
- the value of the cubic term can be used to determine the *size* of the fault.

30. Pseudo-Random Signals Are Difficult to Generate

- In practice, it is difficult to generate pseudo-random signals.
- It is therefore desirable to confirm that non-linearity can be also observed for simpler signals, e.g., for *sinusoid* signals.
- In our experiment, as a signal x_j, we sent an ultrasound wave. To generate this wave:
  - a sinusoid electric signal $x(t) = A \cdot \cos(\omega \cdot t)$ was sent to the transducer,
  - which then generated an ultrasonic wave in the tested object.
- The transducer was set at an angle of incidence of 31° with the plate, so that a wave would go along the surface of the plate (such waves are called *Lamb waves*).
- The transducer is somewhat non-linear.
- To separate the non-linearity of the transducer from the non-linearity of the plate itself, we placed *two* sensors on the plate:
  - the first sensor is located near the transducer, and it measures the ultrasonic wave $x_1(t)$ that the transducer generates;
  - the second sensor is located at a distance from the transducer, and it measure the wave $x_2(t)$ changed after passing through the plate.
- Then, we check whether $x_2(t)$ linearly depends on $x_1(t)$.

31. Experiment with Sinusoid Signals: A Software Part

- After measuring the two signals $x_1(t)$ and $x_2(t)$, we apply FFT to both.
- Compute the total energy $E_1 = \int |\hat{x}_1(\omega)|^2 \, d\omega$ of the signal x_1 in the frequency range
  $[350\text{ KHz}, 650\text{Khz}]$ of the transducer.
- Compute the total energy $E_2 = \int |\hat{x}_2(\omega)|^2 \, d\omega$ of the signal x_2 in the same frequency range.
- Check whether E_2 is a linear function of E_1, i.e., whether there exist k and n for which
$$E_2 = k \cdot E_1 + n.$$
- Due to inevitable measurement inaccuracy, after each measurement, we do not get the *exact* values $E_i(V)$.
- We only get an *interval* $[E_i^-(V), E_i^+(V)]$ of possible values of $E_i(V)$.
- The question is: is this data consistent with the assumption that $E_2(V)$ is a linear function of $E_1(V)$?
- In other words, it is possible to find real numbers $k > 0$, n, and values $E_1(V) \in [E_1^-(V), E_1^+(V)]$ and $E_2(V) \in [E_2^-(V), E_2^+(V)]$ for which
$$E_2(V) = k \cdot E_1(V) + n?$$

32. Taking Measurement Inaccuracy into Consideration: Solution

- **Proposition:** *The set of intervals* $[E_1^-(V), E_1^+(V)]$, $[E_2^-(V), E_2^+(V)]$ *is consistent with the assumption that* $E_2(V)$ *is a linear function of* $E_1(V)$ *if and only if the following inequality is true:*

$$\max_{V'<V} \frac{E_2^-(V) - E_2^+(V')}{E_1^+(V) - E_1^-(V')} \le \max_{V<V'} \frac{E_2^+(V') - E_2^-(V)}{E_1^-(V') - E_1^+(V)}.$$

- **Algorithm:** To check non-linearity, we must check the above inequality.
- *Practical recommendation: brief summary:* To detect the faults, we must use at least *two* different signal levels.
  - If the increase in the signal level x_j leads to a *proportional* increase in the measured values y_i, then most probably the object is *smooth*.
  - If the dependence of y_i on x_j is *non-linear*, then, most probably, there is a *fault*, so further analysis is needed.

33. Experimental Results

- *Undamaged case:*

V	$[E_1^-(V), E_1^+(V)]$ undamaged, 10^6	$[E_2^-(V), E_2^+(V)]$ undamaged, 10^6
0V	$[0.00, 0.01]$	$[0.00, 0.01]$
6V	$[2.65, 2.66]$	$[1.59, 1.61]$
7V	$[3.12, 3.14]$	$[1.86, 1.88]$
8V	$[3.62, 3.64]$	$[2.16, 2.18]$
9V	$[4.59, 4.69]$	$[2.70, 2.80]$

In the undamaged case, we clearly have a linear dependency:

$$E_2(V) \approx 0.6 \cdot E_1(V).$$

- *Damaged case:*

V	$[E_1^-(V), E_1^+(V)]$ damaged, 10^5	$[E_2^-(V), E_2^+(V)]$ damaged, 10^4
0V	$[0.02, 0.03]$	$[0.06, 0.11]$
6V	$[0.69, 0.70]$	$[0.23, 0.28]$
7V	$[0.87, 0.92]$	$[0.14, 0.23]$
8V	$[1.05, 1.08]$	$[4.75, 4.84]$
9V	$[1.28, 1.32]$	$[5.57, 5.80]$

In the damaged case, the dependence is clearly non-linear.