How the Proportion of People Who Agree to Perform a Task Depends on the Stimulus: A Theoretical Explanation of the Empirical Formula

Laxman Bokati[1], Vladik Krenovich[1], and Doan Thanh Ha[2]

[1]Computational Science Program
University of Texas at El Paso
El Paso, Texas 79968, USA
laxman@miners.utep.edu, vladik@utep.edu
[2]Banking University of Ho Chi Minh City
36 Ton That Dam, District 1
Ho Chi Minh City, Vietnam, hadt@buh.edu.vn

1. The Larger the Stimulus, the More People Agree to Do the Task

- In economics, we need to entice people to perform certain tasks – whether it is
  - planting crops
  - or working on a factory
  - or writing a software package.
- When the corresponding stimulus is too small, no one will agree to perform the task.
- When the stimulus is very high, everyone will agree.
- The proportion p of people who agree to perform a task will increase with the increase in the stimulus s.

2. It Is Desirable to Know the Exact Amount of Stimulus

- A company wants certain tasks to be performed, so it has to use some stimulus.
- It is therefore desirable to find the exact amount of stimulus needed:
  - if the stimulus is too low, no one will volunteer,
  - if it is very high, the tasks will be performed, but the company will lose too much money.

3. How the Amount of Stimulus Is Usually Determined Now

- In many cases, the selection of the right stimulus is done mostly by trial and error.
- This is, e.g., how airline companies, in an overbooked situation, ask for volunteers to give up their seats.
- They increase the award offered to potential volunteers until they get enough volunteers.

4. Formulas Are Needed, And There Are Such Formulas

- Trial-and-error is a lengthy process, difficult to predict.
- It is therefore desirable to have an expressions helping us select the right amount of stimulus.
- Such expressions exist.
- The most empirically adequate expression is $p = \dfrac{s^q}{s^q + c}$ for some constants q and c.

5. This Formula Is Purely Empirical

- One of the main limitation of this formula is that:
  - it is purely empirical,
  - it does not have a convincing theoretical explanation.
- Practitioners are usually very suspicious of best-fit purely empirical formulas.
- They are reluctant so use these formulas.
- They prefer formulas for which some theoretical explanation exists.
- Indeed, empirical formulas often turn out to be wrong.
- And in economics and related areas, such later-wrong empirical formulas are ubiquitous.
- When a country has a boom, empirical formulas predict exponential growth forever.

6. This Formula Is Purely Empirical (cont-d)

- When, in the 1920s, the number of telephone operators started growing exponentially:
  - empirical formulas predicted that in a few decades,
  - half of the population will be telephone operators.
- There are many examples like that.
- It is thus desirable to come up with a theoretical explanation for empirical formulas.
- In this talk, we provide a theoretical explanation for the above formula.

7. Let Us Reformulate the Problem in Terms of Probabilities

- In the above text, we talked about proportion of people who take on the task.
- From the mathematical viewpoint, a proportion is not something about which we know much.
- But what is proportion?
- It is simply the probability that a randomly selected person will take on the task.
- So, whatever we said about proportions can be reformulated in terms of probabilities.
- And about probabilities, we know a lot!

8. What Do We Know About Probabilities?

- One of the most widely used facts about probabilities is that:
  - if we add new evidence E,
  - the probability of each hypothesis H_i changes according to the Bayes formula.
- Namely, it changes from the original value $p_0(H_i)$ to the new value

$$p(H_i \mid E) = \frac{p_0(H_i) \cdot p(E \mid H_i)}{\sum\limits_j p_0(H_j) \cdot p(E \mid H_j)}.$$

- In our case, we have two hypotheses:
  - the hypothesis H_0 that the person will take on the task whose probability is $p(H_0)$, and
  - the hypothesis H_1 that the person will not take on the task; its probability is $p(H_1) = 1 - p(H_0)$.

9. What We Know About Probabilities (cont-d)

- In this case, the Bayes formula takes the form

$$p(H_0\,|\,E) = \frac{p_0(H_0)\cdot p(E\,|\,H_0)}{p_0(H_0)\cdot p(E\,|\,H_0) + (1-p_0(H_0))\cdot p(E\,|\,H_1)} =$$

$$\frac{p(H_0)\cdot p(E\,|\,H_0)}{p(H_0)\cdot (p(E\,|\,H_0) - p(E\,|\,H_1)) + p(E\,|\,H_1)},\ \text{i.e.}$$

$$p' = \frac{p\cdot p(E\,|\,H_0)}{p\cdot (p(E\,|\,H_0) - p(E\,|\,H_1)) + p(E\,|\,H_1)}.$$

- Here, we denoted $p = p_0(H_0)$ and $p' = p(H_0\,|\,E)$.

- If we divide both the numerator and the denominator of this formula by $p(E\,|\,H_1)$, we get:

$$p' = \frac{p\cdot \dfrac{p(E\,|\,H_0)}{p(E\,|\,H_1)}}{1 + p\cdot\left(\dfrac{p(E\,|\,H_0)}{p(E\,|\,H_1)} - 1\right)}, = \frac{a\cdot p}{1 + (1-a)\cdot p}.$$

10. What We Know About Probabilities (cont-d)

- Here, we denoted $a \stackrel{\text{def}}{=} \dfrac{p(E\,|\,H_0)}{p(E\,|\,H_1)}$.
- In other words:
  - the change of the probability from the previous value p to the new value p'
  - is described by a fractional-linear formula.
- Our idea is that:
  - when we increase the stimulus,
  - the resulting change of the probability should follow such a formula.

11. How Can We Formalize This Idea

- What does it mean "increase the stimulus"?
- Intuitively, it means that we increase all the previous stimuli the same way.
- What does that mean?
- If we add \$10 to all the stimulus values, this does not mean that we increases all the stimuli the same way.
- For example:
  - if the previous stimulus was \$5, this is a drastic 3-times increase, but
  - if the previous stimulus was \$1000, this is a barely noticeable 1% increase.

12. How Can We Formalize This Idea (cont-d)

- From the economic viewpoint, it makes more sense to increase all the stimulus values proportionally; e.g.:
  - increase all the values by 1%, or
  - increase all the values by 10%, or
  - increase all the values by a factor of three.
- With such an increase, instead of previous stimulus value s, we get a new stimulus value $\lambda \cdot s$, where, e.g.:
  - an over-the-board 1% increase means $\lambda = 1.01$,
  - an over-the-board 10% increase means $\lambda = 1.1$, and
  - an over-the-board 3-times increase means $\lambda = 3$.
- In these terms, the main idea takes the following form.

13. Resulting Formulation and the Main Result

- We want to find an increasing function $p(s)$ for which $p(0) = 0$, $p(s) \to 1$ as $s \to \infty$, and:
  - for every $\lambda > 0$,
  - there exists $a(\lambda)$ for which, for all s, we have

$$p(\lambda \cdot s) = \frac{a(\lambda) \cdot p(s)}{1 + (a(\lambda) - 1) \cdot p(s)}.$$

- **Proposition.** *Every function $p(s)$ satisfying these conditions has the form $p = \dfrac{s^q}{s^q + c}$, for some q and c.*
- Thus, we indeed have the desired justification of the empirical formula.

14. Proof: Let Us Reformulate the Bayes Formula in Terms of Odds

- For this proof, it is convenient to reformulate probabilities p in terms of the *odds* $o = \dfrac{p}{1-p}$.
- Let us first find the odds corresponding to the new probability $p(\lambda \cdot s)$.
- From the above formula, we get

$$1 - p(\lambda \cdot s) = 1 - \frac{a(\lambda) \cdot p(s)}{1 + (a(\lambda) - 1) \cdot p(s)} =$$

$$\frac{1 + a(\lambda) \cdot p(s) - p(s) - a(\lambda) \cdot p(s)}{1 + (a(\lambda) - 1) \cdot p(s)} = \frac{1 - p(s)}{1 + (a(\lambda) - 1) \cdot p(s)}.$$

15. Bayes Formula (cont-d)

- Dividing $p(\lambda \cdot s)$ by $1 - p(\lambda \cdot s)$, we get

$$o(\lambda \cdot s) = \frac{p(\lambda \cdot s)}{1 - p(\lambda \cdot s)} = \frac{a(\lambda) \cdot p(s)}{1 - p(s)} = a(\lambda) \cdot \frac{p(s)}{1 - p(s)}.$$

- The ratio in the right-hand side is exactly the odds $o(s)$ corresponding to the probability $p(s)$.
- So, we conclude that: $o(\lambda \cdot s) = a(\lambda) \cdot o(s)$.

16. Now, We Can Use the Known Solution to the Functional Equation

- We know that $o(\lambda \cdot s) = a(\lambda) \cdot o(s)$.
- It is known that every monotonic solution of this equation has the form $o(s) = C \cdot s^q$ for some C and q.
- The general proof of this statement is somewhat complicated.
- However, it becomes very straightforward if assume that $p(s)$ is differentiable.
- In this case, the ratio $o(s)$ is also differentiable.
- $a(\lambda) = \dfrac{o(\lambda \cdot s)}{o(s)}$ is the ratio of two differentiable functions, hence also differentiable.
- Thus, we can differentiate both sides of our equation with respect to λ and get $s \cdot o'(\lambda \cdot s) = a'(\lambda) \cdot o(s)$.

17. Solving Functional Equation (cont-d)

- In particular, for $\lambda = 1$, we get $s \cdot o'(s) = q \cdot o(s)$, where we denoted $q \stackrel{\text{def}}{=} a'(1)$.
- In other words, we have $s \cdot \dfrac{do}{ds} = q \cdot o$.
- We can separate s and o if we divide both sides by $s \cdot o$ and multiply by ds: $\dfrac{do}{o} = q \cdot \dfrac{ds}{s}$.
- Integrating both sides, we get $\ln(o) = q \cdot \ln(s) + C_0$, where C_0 is an integration constant.
- By applying $\exp(x)$ to both sides, we then get $o(s) = C \cdot s^q$, where we denoted $C \stackrel{\text{def}}{=} \exp(C_0)$.

18. Deriving the Desired Formula

- We have $o(s) = \dfrac{p(s)}{1 - p(s)} = C \cdot s^q$.
- By taking the inverse of both sides, we get

$$\frac{1 - p(s)}{p(s)} = 1 - \frac{1}{p(s)} = C^{-1} \cdot s^{-q}.$$

- Thus $\dfrac{1}{p(s)} = 1 - C^{-1} \cdot s^{-q}$ and $p(s) = \dfrac{1}{1 - C^{-1} \cdot s^{-q}}$.
- Multiplying both the numerator and the denominator by s^q, we get $p(s) = \dfrac{s^q}{s^q - C^{-1}}$.
- Probabilities are always smaller than or equal to 1, thus $s^q \le s^q - C^{-1}$, i.e., $C^{-1} < 0$.
- For $c \stackrel{\text{def}}{=} -C^{-1}$, we get the desired formula $p = \dfrac{s^q}{s^q + c}$.
- The main result is thus proven.

19. Acknowledgments

This work was supported in part by the National Science Foundation grants:

- 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science),
- HRD-1242122 (Cyber-ShARE Center of Excellence).