

Distributions on an Interval as a Scale-Invariant Combination of Scale-Invariant Functions: Theoretical Explanation of Empirical Marchenko-Pastur-Type Distributions

Vladik Kreinovich¹, Kevin Alvarez¹, and Chon Van Le²

¹Department of Computer Science, University of Texas at El Paso
500 W. University, El Paso, Texas 79968, USA
vladik@utep.edu, kalvarez9@miners.utep.edu

²International University of Ho-Chi-Minh City, Vietnam
lvchon@hcmiu.edu.vn

Some Distributions ...

Empirical Fact

Scale-Invariance: A ...

Which Functions Are ...

Starting Point Can ...

What Are Natural ...

How Can We Use ...

Which Combination ...

Resulting Expression

Home Page

Title Page

⏪

⏩

◀

▶

Page 1 of 14

Go Back

Full Screen

Close

Quit

1. Some Distributions Are Located on an Interval

- For many physical and economic quantities x :
 - there are is a lower bound \underline{x} on its possible values and
 - there is an upper bound \bar{x} on its possible values.
- This means that all possible values on the quantity x are located on the interval $[\underline{x}, \bar{x}]$.
- In particular, this means that the probability distribution of this quantity is located on this interval.

Empirical Fact

Scale-Invariance: A . . .

Which Functions Are . . .

Starting Point Can . . .

What Are Natural . . .

How Can We Use . . .

Which Combination . . .

Resulting Expression

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Page 2 of 14

Go Back

Full Screen

Close

Quit

2. Empirical Fact

- In economics, many such distributions have the form

$$\rho(x) = \text{const} \cdot (x - \underline{x})^{\alpha_-} \cdot (\bar{x} - x)^{\alpha_+} \cdot x^\alpha.$$

- In particular:
 - for $\alpha_- = \alpha_+ = 0.5$ and $\alpha = -1$,
 - we get the Marchenko-Pastur distribution – the distribution of eigenvalues of a random matrix.
- For example, this is how the eigenvalues of the cross-correlation matrix of different stocks are distributed.
- However, in other cases, we have distributions with different values of α_- , α_+ , and α .
- How can we explain the ubiquity of such Marchenko-Pastur-type distributions?
- In this talk, we use the idea of scale invariance to provide a theoretical explanation for these distributions.

3. Scale-Invariance: A Brief Reminder

- The numerical value of a quantity depends on the choice of a measuring unit.
- For example, we can describe the price of a financial instrument in Euros, in US Dollars, in Japanese Yen.
- The instrument is the same in all cases, but:
 - for different currencies,
 - we will get different numerical representations of the same price.
- In many situations:
 - there is no reason to select this or that measuring unit,
 - the choice of the unit is just a matter of convention.

4. Scale-Invariance (cont-d)

- In such situations, it makes sense to require that:
 - the formula $y = f(x)$ describing the dependence between quantities x and y
 - should not change if we replace the original measuring unit for x .
- Here:
 - if we replace the original measuring unit for x by a new unit which is λ times smaller,
 - then all the numerical values of this quantity will be multiplied by λ : $x \rightarrow x' = \lambda \cdot x$.
- Of course:
 - for the formula $y = f(x)$ to be valid in the new units,
 - we need to appropriately change the unit for y .

Some Distributions ...

Empirical Fact

Scale-Invariance: A ...

Which Functions Are ...

Starting Point Can ...

What Are Natural ...

How Can We Use ...

Which Combination ...

Resulting Expression

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 5 of 14

Go Back

Full Screen

Close

Quit

5. Scale-Invariance (cont-d)

- For example:
 - the formula $y = x^2$ that describes how the area of a square depends on its size
 - does not depend on the choice of units.
- However:
 - if we replace, e.g., meters with centimeters,
 - we need to also replace square meters with square centimeters.
- In general, scale-invariance of a function $f(x)$ takes the following form:

*For every $\lambda > 0$, there exists a value $\mu(\lambda) > 0$
for which $y = f(x)$ implies $y' = f(x')$,
where $x' \stackrel{\text{def}}{=} \lambda \cdot x$ and $y' \stackrel{\text{def}}{=} \mu(\lambda) \cdot y$.*

Some Distributions ...

Empirical Fact

Scale-Invariance: A ...

Which Functions Are ...

Starting Point Can ...

What Are Natural ...

How Can We Use ...

Which Combination ...

Resulting Expression

Home Page

Title Page



Page 6 of 14

Go Back

Full Screen

Close

Quit

6. Which Functions Are Scale-Invariant?

- Substituting the expressions $x' = \lambda \cdot x$ and $y' = \mu(\lambda) \cdot y$ into the formula $y' = f(x')$, we get $\mu(\lambda) \cdot y = f(\lambda \cdot x)$.
- Here, $y = f(x)$, so we get

$$f(\lambda \cdot x) = \mu(\lambda) \cdot f(x).$$

- It is known that:
 - every continuous (and even every measurable) f-n $f(x)$ that satisfies this equation for all x and λ
 - has the form $f(x) = c \cdot x^a$, for some constants c and a .

7. Starting Point Can Also Be Different

- For many quantities – e.g., for time – we can also select different starting points:
 - if we replace the original starting point with a new starting point which is x_0 units before,
 - then all the numerical values x of this quantity are replaced by new values: $x' = x + x_0$.
- In particular:
 - if we have a scale-invariant dependence $f(x) = c \cdot (x')^a$ in the new scale,
 - then, in the old scale, this dependence takes the form $y = c \cdot (x + x_0)^a$.

Some Distributions ...

Empirical Fact

Scale-Invariance: A ...

Which Functions Are ...

Starting Point Can ...

What Are Natural ...

How Can We Use ...

Which Combination ...

Resulting Expression

Home Page

Title Page



Page 8 of 14

Go Back

Full Screen

Close

Quit

8. What Are Natural Starting Points for Functions Located on an Interval $[\underline{x}, \bar{x}]$

- If we know that a quantity x is always located on an interval $[\underline{x}, \bar{x}]$, then we have two natural starting points:

\underline{x} and \bar{x} .

- Thus, in addition to the original scale-invariant functions $f(x) = c \cdot x^a$, we also get functions

$$f(x) = c_- \cdot (x - \underline{x})^{a_-} \text{ and } f(x) = c_+ \cdot (\bar{x} - x)^{a_+}.$$

- Since we need a single function, we need to combine these functions.

Some Distributions ...

Empirical Fact

Scale-Invariance: A ...

Which Functions Are ...

Starting Point Can ...

What Are Natural ...

How Can We Use ...

Which Combination ...

Resulting Expression

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 9 of 14

Go Back

Full Screen

Close

Quit

9. How Can We Use Scale-Invariance to Combine Different Functions?

- We want to combine several functions $y_1 = f_1(x), \dots, y_n = f_n(x)$ into a single quantity $y = F(y_1, \dots, y_n)$.
- In view of the above, it makes sense to do it in scale-invariant way.
- In other words, we want to find a function $F(y_1, \dots, y_n)$ that has the following property:
- *For every combination of possible values $\lambda_1 > 0, \dots, \lambda_n > 0$, there exists $\mu(\lambda_1, \dots, \lambda_n)$ for which:*
 - $y = F(y_1, \dots, y_n)$ implies that $y' = F(y'_1, \dots, y'_n)$,
 - where $y'_1 \stackrel{\text{def}}{=} \lambda_1 \cdot y_1, \dots, y'_n \stackrel{\text{def}}{=} \lambda_n \cdot y_n$, and $y' \stackrel{\text{def}}{=} \mu(\lambda_1, \dots, \lambda_n) \cdot y$.

10. Which Combination Operations Are Scale-Invariant?

- Substituting $y'_i = \lambda_i \cdot y_i$ and $y' = \mu(\lambda_1, \dots, \lambda_n) \cdot y$ into the formula $y' = F(y'_1, \dots, y'_n)$, we get

$$\mu(\lambda_1, \dots, \lambda_n) \cdot y = F(\lambda_1 \cdot y_1, \dots, \lambda_n \cdot y_n).$$

- Here, $y = F(y_1, \dots, y_n)$, so we get

$$F(\lambda_1 \cdot y_1, \dots, \lambda_n \cdot y_n) = \mu(\lambda_1, \dots, \lambda_n) \cdot F(y_1, \dots, y_n).$$

- It is known that:

- every continuous function $F(y_1, \dots, y_n)$ that satisfies this equation for all y_i and λ_i
- has the form

$$y = F(y_1, \dots, y_n) = C \cdot y_1^{a_1} \cdot \dots \cdot y_n^{a_n}, \text{ for some } C \text{ and } a_i.$$

11. Resulting Expression

- In our case, we combine three expressions:
 - the expression for $y_1 = c \cdot x^a$,
 - the expression for $y_2 = c_- \cdot (x - \underline{x})^{\alpha_-}$, and
 - the expression for $y_3 = c_+ \cdot (x - \bar{x})^{\alpha_+}$.
- Substituting the expressions for y_i into the formula $y = C \cdot y_1^{\alpha_1} \cdot y_2^{\alpha_2} \cdot y_3^{\alpha_3}$, we conclude that

$$y = c_0 \cdot (x - \underline{x})^{\alpha_-} \cdot (\bar{x} - x)^{\alpha_+} \cdot x^{\alpha}.$$

- Here, $c_0 \stackrel{\text{def}}{=} C \cdot c^{\alpha_1} \cdot c_-^{\alpha_2} \cdot c_+^{\alpha_3}$, $\alpha_- \stackrel{\text{def}}{=} \alpha_2 \cdot a_-$, $\alpha_+ \stackrel{\text{def}}{=} \alpha_3 \cdot a_+$, and $\alpha \stackrel{\text{def}}{=} \alpha_1 \cdot a$.
- For the case when y is the probability density, this is exactly the desired formula.
- So, we have indeed explained the empirical formula by using scale-invariance.

12. Comment

- Our explanation is more general than explaining the empirical distributions.
- It also explains, e.g., why in many cases,
 - Bernstein polynomials, i.e., sums of monomials of the type $(x - \underline{x})^{a_-} \cdot (\bar{x} - x)^{a_+}$,
 - provide a good approximation to functions located on an interval.

Some Distributions ...

Empirical Fact

Scale-Invariance: A ...

Which Functions Are ...

Starting Point Can ...

What Are Natural ...

How Can We Use ...

Which Combination ...

Resulting Expression

Home Page

Title Page



Page 13 of 14

Go Back

Full Screen

Close

Quit

13. Acknowledgments

This work was supported in part by the National Science Foundation grants:

- 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science);
- HRD-1834620 and HRD-2034030 (CAHSI Includes).

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.

Some Distributions . . .

Empirical Fact

Scale-Invariance: A . . .

Which Functions Are . . .

Starting Point Can . . .

What Are Natural . . .

How Can We Use . . .

Which Combination . . .

Resulting Expression

[Home Page](#)

[Title Page](#)



Page 14 of 14

[Go Back](#)

[Full Screen](#)

[Close](#)

[Quit](#)