Correcting Interval-Valued Expert Estimates: Empirical Formulas Explained

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1. Need for interval estimates

- In economics – as in many other application areas – we often rely on expert estimates.
- Of course, expert estimates are approximate.
- So, to be able to use them effectively, we need to know how accurate they are.
- There are two main idea on how to gauge the accuracy of an expert estimate.
- The first idea is to ask the expert him/herself to gauge this accuracy.
- Namely, to ask the expert to provide an *interval* of possible values instead of a single value.

2. Need for interval estimates (cont-d)

- The second idea is:
  - to ask one or more other experts, and
  - to consider the range formed by their values as the reasonable range of possible values for the corresponding quantity.
- For example, if one experts predicts the value 15, and the two others predict 10 and 20, then we take the interval $[10, 20]$.

3. Need for correction

- When an expert provides an interval of possible values, a correction may be needed.
- Sometimes, the expert is too confident, and the interval provided by the expert is too narrow.
- In this case, a natural idea is to widen it.
- Sometimes, the expert is too cautious, and the interval provided by the expert is too wide.
- In this case, a natural idea is to make it narrower.

4. Need for correction (cont-d)

- Similarly, when we have an interval range formed by numerical estimates made by different experts, a correction may be needed.
- Sometimes, the experts' estimates are too close to each other, so the resulting interval is too narrow.
- In this case, a natural idea is to widen it.
- Sometimes, the experts' estimates are too far away from each other.
- For example, one expert's predictions are too optimistic, and another expert's predictions are too pessimistic.
- In this case, the resulting interval is too wide, so a natural idea is to make it narrower.

5. How do people correct the corresponding intervals?

- Empirical data shows the corrected version $[A, B]$ of the original interval $[a, b]$ usually follows the formula

$$[A, B] = \left[a \cdot \frac{1+\alpha}{2} + b \cdot \frac{1-\alpha}{2}, a \cdot \frac{1-\alpha}{2} + b \cdot \frac{1+\alpha}{2}\right] \text{ for some } \alpha > 0.$$

- The values $\alpha < 1$ correspond to shrinking.
- The values $\alpha > 1$ correspond to stretching of the interval.
- This formula was first proposed by Gajdos in 2013 for expert estimates of probabilities.
- In Smithson (2013) this formula was extended to general (not necessarily probabilistic) expert estimates.
- In this talk, we provide a possible explanation for this formula.

6. Analysis of the problem

- We need to find algorithms $A(a, b)$ and $B(a, b)$ that, given the expert-provided values a and b, produced the corrected values A and B.
- Let us analyze what are the natural properties of these algorithms.

7. The correction formula should not depend on the monetary unit

- The same financial predictions can be described in different monetary units.
- For example, predictions related to Mexican economy can be made in Mexican pesos or in US dollars.
- We are looking for general correction formulas, formulas that would be applicable to all possible interval-valued expert estimates.
- Suppose that we first applied this formula to the interval $[a,b]$ described in one monetary units.
- Then, in these units, the corrected interval takes the form $[A(a,b),B(a,b)]$.

8. The correction formula should not depend on the monetary unit (cont-d)

- Another possibility is:
  - to first translate into a different monetary unit,
  - to make a correction there, and then
  - to translate the result back into the original monetary unit.
- If we select a different monetary unit which is λ times smaller than the original one, then all numerical values multiply by λ.
- In particular, in the new units, the original interval $[a, b]$ will take the form $[\lambda \cdot a, \lambda \cdot b]$.
- If we apply the same correction algorithm to this interval, we get – in the new units – the following corrected interval:

$$[A(\lambda \cdot a, \lambda \cdot b), B(\lambda \cdot a, \lambda \cdot b)].$$

- To describe this corrected interval in the original units, we need to divide both its endpoints by λ.

9. The correction formula should not depend on the monetary unit (cont-d)

- As a result, we get the corrected interval expressed in the original units:
$$\left[\frac{1}{\lambda}\cdot A(\lambda\cdot a,\lambda\cdot b),\frac{1}{\lambda}\cdot B(\lambda\cdot a,\lambda\cdot b)\right].$$

- It is reasonable to require that the corrected interval should be the same whether we use the original monetary units or different units:
$$[A(a,b),B(a,b)]=\left[\frac{1}{\lambda}\cdot A(\lambda\cdot a,\lambda\cdot b),\frac{1}{\lambda}\cdot B(\lambda\cdot a,\lambda\cdot b)\right],\ \text{i.e.:}$$
$$A(a,b)=\frac{1}{\lambda}\cdot A(\lambda\cdot a,\lambda\cdot b)\text{ and }B(a,b)=\frac{1}{\lambda}\cdot B(\lambda\cdot a,\lambda\cdot b).$$

- This property is known as *scale-invariance*.

10. Shift-invariance

- Suppose that the expected company's income consists of:
  - the fixed amount f – e.g., determined by the current contracts – and
  - some additional amount x that will depend on the relation between supply and demand.
- Suppose that the expert predicts this additional amount to be somewhere in the interval $[a, b]$.
- This means that the overall company's income is predicted to be between $f + a$ and $f + b$, i.e., somewhere in the interval $[f + a, f + b]$.
- If we believe that the expert estimate needs corrections, then we have two possible ways to perform this correction.
- We can apply the correction to the original interval $[a, b]$.
- This results in the corrected interval estimate $[A(a, b), B(a, b)]$ for the additional income.

11. Shift-invariance (cont-d)

- In this case, the interval estimate for the overall income will be

$$[f + A(a, b), f + B(a, b)].$$

- Alternatively, we can apply the correction to the interval $[f+a, f+b]$ describing the overall income.
- In this case, the resulting corrected interval for the overall income will have the form $[A(f + a, f + b), B(f + a, f + b)]$.
- It is reasonable to require that the two methods should lead to the exact same interval estimate for the overall income:

$$[f + A(a, b), f + B(a, b)] = [A(f + a, f + b), B(f + a, f + b)], \text{ i.e.:}$$

$$f + A(a, b) = A(f + a, f + b) \text{ and } f + B(a, b) = B(f + a, f + b).$$

- This property is known as *shift-invariance*.

12. Sign invariance

- One of the possible expert predictions is, e.g., how much bank B_1 will owe a bank B_2 at a certain future date.
- This amount can be positive – meaning that the bank B_1 will owe some money to the bank B_2.
- This amount can also be negative – meaning that, according to the expert, the bank B_2 will owe money to the bank B_1.
- Suppose that the expert estimates this amount by an interval $[a, b]$.
- This means that:
  - if we ask the same expert a different question: how much money will the bank B_2 owe to the bank B_1,
  - this expert will provide the interval $[-b, -a]$, i.e., the set of all the values $-x$ when $x \in [a, b]$.
- In this case, we also have two possible ways to perform this correction.

13. Sign invariance (cont-d)

- We can apply the correction to the original interval $[a, b]$, resulting in the corrected interval estimate $[A(a, b), B(a, b)]$.
- Alternatively, we can:
  - apply the correction to the interval $[-b, -a]$ describing how much the bank B_2 will owe to the bank B_1, and get the corrected interval
$$[A(-b, -a), B(-b, -a)];$$
  - by changing the sign, we get an interval estimate of how much the bank B_1 will owe to the bank B_2: $[-B(-b, -a), -A(-b, -a)]$.
- It is reasonable to require that the two methods should lead to the exact same interval estimate for the overall amount:
$$[A(a, b), B(a, b)] = [-B(-b, -a), -A(-b, -a)], \text{ i.e.:}$$
$$A(a, b) = -B(-b, -a) \text{ and } B(a, b) = -A(-b, -a).$$
- We will call this property *sign-invariance.*

14. Main Result and Its Proof

- **Proposition.** *Every scale-, shift-, and sign-invariant transformation has the form*

$$[A, B] = \left[a \cdot \frac{1+\alpha}{2} + b \cdot \frac{1-\alpha}{2}, a \cdot \frac{1-\alpha}{2} + b \cdot \frac{1+\alpha}{2}\right] \text{ for some } \alpha > 0.$$

- **Proof.** Indeed, due to shift-invariance for $f = a$, we have

$$B(a, b) = a + B(0, b-a).$$

- Due to scale-invariance for $\lambda = b - a$, we have

$$B(0, 1) = \frac{1}{b-a} \cdot B(0, b-a), \text{ hence } B(0, b-a) = (b-a) \cdot B(0, 1).$$

- Let us denote $\alpha \stackrel{\text{def}}{=} 2B(0, 1) - 1$, then $B(0, 1) = \dfrac{1+\alpha}{2}$, so

$$B(0, b-a) = (b-a) \cdot \frac{1+\alpha}{2} = b \cdot \frac{1+\alpha}{2} - a \cdot \frac{1+\alpha}{2}.$$

15. Main Result and Its Proof (cont-d)

- Substituting this expression into the formula for $B(a,b)$, we get

$$B(a,b) = a + b \cdot \frac{1+\alpha}{2} - a \cdot \frac{1+\alpha}{2} = a \cdot \frac{1-\alpha}{2} + b \cdot \frac{1+\alpha}{2}.$$

- This is exactly the expression for $B(a,b)$ that we want to explain.
- Now, by using sign-invariance, we conclude that

$$A(a,b) = -B(-b,-a) = -\left((-b) \cdot \frac{1-\alpha}{2} + (-a) \cdot \frac{1+\alpha}{2}\right) =$$

$$a \cdot \frac{1+\alpha}{2} + b \cdot \frac{1-\alpha}{2}.$$

- This is also exactly the expression for $A(a,b)$ corresponding to the desired formula.
- Thus, the proposition is proven.

16. Comment

- The paper Smithson (2015) also mentions an additional empirical fact that it find difficult to explain: that
  - if two experts provide interval estimates,
  - people put trust into pairs of intervals that have comparable width.
- From our viewpoint, this is easy to explain – if two experts, based on the same data:
  - provide completely different estimates of how accurately we can make a prediction based on this data,
  - then we do not trust any of these experts.
- This is one of the cases when the two supposed experts are inconsistent.
- A similar phenomenon happens when the two experts provide drastically different numbers; then, we do not trust either of them.

17. Bibliography

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18. Acknowledgments

- This work was supported in part by the National Science Foundation grants:
  - 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and
  - HRD-1834620 and HRD-2034030 (CAHSI Includes).
- It was also supported by the AT&T Fellowship in Information Technology.
- It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.